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Semiparametric proper efficiency principles and duality models for constrained multiobjective fractional optimal control problems containing arbitrary norms

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Abstract

Semiparametric necessary and sufficient proper efficiency conditions are established for a class of constrained multiobjective fractional optimal control problems with linear dynamics, containing arbitrary norms. Moreover, utilizing these proper efficiency results, eight semiparametric duality models are formulated and appropriate duality theorems are proved. These proper efficiency and duality criteria contain, as special cases, similar results for several classes of unorthodox optimal control problems with multiple, fractional, and conventional objective functions, which are particular cases of the main problem considered in this paper.

1. Introduction

In this paper, we shall establish semiparametric necessary and sufficient proper efficiency conditions and duality results for the following constrained multiobjective fractional optimal control problem containing arbitrary norms:

$$\text{(P)} \quad \text{Minimize} \left(\frac{\int_a^b [f_1(x(t), u(t), t) + \|K_1(t)x(t)\|_{k(1)} + \|L_1(t)u(t)\|_{\ell(1)}]\,dt}{\int_a^b [g_1(x(t), u(t), t) - \|M_1(t)x(t)\|_{m(1)} - \|N_1(t)u(t)\|_{n(1)}]\,dt}, \right.$$

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$$\left. \ldots, \frac{\int_a^b [f_r(x(t), u(t), t) + \|K_r(t)x(t)\|_{k(r)} + \|L_r(t)u(t)\|_{\ell(r)}]\,dt}{\int_a^b [g_r(x(t), u(t), t) - \|M_r(t)x(t)\|_{m(r)} - \|N_r(t)u(t)\|_{n(r)}]\,dt} \right)^T$$

subject to

$$x(a) = 0, \qquad x(b) = 0, \tag{1.1}$$

$$Dx(t) = A(t)x(t) + B(t)u(t), \quad t \in [a, b], \tag{1.2}$$

$$h_j\big(x(t), u(t), t\big) + \big\|P_j(t)x(t)\big\|_{p(j)} + \big\|Q_j(t)u(t)\big\|_{q(j)} \leqq 0,$$
$$t \in [a, b], \ j \in \underline{s}, \tag{1.3}$$

$$x \in C^n[a, b], \qquad u \in \mathrm{PWS}^m[a, b],$$

where $C^n[a, b]$ is the space of all continuous n-dimensional vector functions $x : [a, b] \to \mathfrak{R}^n$ (n-dimensional Euclidean space) defined on the compact interval $[a, b]$ of the real line $\mathfrak{R}$, with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is defined by

$$y = Dx \quad \Leftrightarrow \quad x(t) = \int_a^t y(\tau)\, Dt;$$

thus $D = d/dt$ except at discontinuities of the piecewise smooth function $y : [a, b] \to \mathfrak{R}^n$; $\mathrm{PWS}^m[a, b]$ is the space of all piecewise smooth m-dimensional vector functions defined on $[a, b]$, with the uniform norm $\|\cdot\|_\infty$; f_i, g_i, $i \in \underline{r} \in \{1, 2, \ldots, r\}$, and h_j, $j \in \underline{s}$, are continuously differentiable real-valued functions defined on $\mathfrak{R}^n \times \mathfrak{R}^m \times [a, b]$; $f_i(.,., t)$, $g_i(.,., t)$, $i \in \underline{r}$, and $h_j(.,., t)$, $j \in \underline{s}$, are convex on $\mathfrak{R}^n \times \mathfrak{R}^m$ throughout $[a, b]$; $A(t)$, $B(t)$, $K_i(t)$, $L_i(t)$, $M_i(t)$, $N_i(t)$, $P_j(t)$, and $Q_j(t)$, $i \in \underline{r}$, $j \in \underline{s}$, are, respectively, $n \times n$, $n \times m$, $k_i \times n$, $l_i \times m$, $m_i \times n$, $n_i \times m$, $p_j \times n$, and $q_j \times m$ matrices whose entries are continuous real-valued functions defined on $[a, b]$; $\|\cdot\|_{k(i)}$, $\|\cdot\|_{\ell(i)}$, $\|\cdot\|_{m(i)}$, $\|\cdot\|_{n(i)}$, $\|\cdot\|_{p(j)}$, and $\|\cdot\|_{q(j)}$, $i \in \underline{r}$, $j \in \underline{s}$, are arbitrary norms; M^T is the transpose of the matrix M, and for each $i \in \underline{r}$,

$$\int_a^b \big[f_i\big(x(t), u(t), t\big) + \big\|K_i(t)x(t)\big\|_{k(i)} + \big\|L_i(t)u(t)\big\|_{\ell(i)}\big]\,dt \geqq 0$$

and

$$\int_a^b \big[g_i\big(x(t), u(t), t\big) - \big\|M_i(t)x(t)\big\|_{m(i)} - \big\|N_i(t)u(t)\big\|_{n(1)}\big]\,dt > 0$$

for all (x, u) satisfying the constraints of (P).

Recently, some parametric proper efficiency and duality results for (P) were presented in [6]. They were derived indirectly with the help of an equivalent nonfractional multiobjective parametric problem, a set of optimality conditions for a related single-objective optimal control problem, and a certain scalarization scheme. As a consequence of employing two auxiliary parametric problems, two sets of parameters were introduced which are present in the statements of all the ensuing proper efficiency and duality results.

In this study, we shall eliminate one of these two sets of parameters and thus formulate some semiparametric proper efficiency principles and duality models for (P). Subsequently, we shall briefly indicate how these results can be modified and restated for a special case of (P) containing square roots of positive semidefinite quadratic forms.

Obviously, all the results established for (P) are also applicable to the following classes of problems with multiple, fractional, and conventional objective functions, which are special cases of (P):

$$\text{(Pl)}\quad \underset{x,u\in\mathcal{F}}{\text{Minimize}} \left(\int_a^b \left[f_1\big(x(t),u(t),t\big) + \left\|K_1(t)x(t)\right\|_{k(1)} + \left\|L_1(t)u(t)\right\|_{\ell(1)}\right] dt,\right.$$

$$\left.\ldots, \int_a^b \left[f_r\big(x(t),u(t),t\big) + \left\|K_r(t)x(t)\right\|_{k(r)} + \left\|L_r(t)u(t)\right\|_{\ell(r)}\right] dt\right)^T,$$

$$\text{(P2)}\quad \underset{x,u\in\mathcal{F}}{\text{Minimize}}\ \frac{\int_a^b [f_1(x(t),u(t),t) + \|K_1(t)x(t)\|_{k(1)} + \|L_1(t)u(t)\|_{\ell(1)}]\,dt}{\int_a^b [g_1(x(t),u(t),t) - \|M_1(t)x(t)\|_{m(1)} - \|N_1(t)u(t)\|_{n(1)}]\,dt},$$

$$\text{(P3)}\quad \underset{x,u\in\mathcal{F}}{\text{Minimize}} \int_a^b \left[f_1\big(x(t),u(t),t\big) + \left\|K_1(t)x(t)\right\|_{k(1)} + \left\|L_1(t)u(t)\right\|_{\ell(1)}\right] dt,$$

where $\mathcal{F}$ (assumed to be nonempty) is the feasible set of (P), that is,

$$\mathcal{F} = \left\{(x,u) \in C^n[a,b] \times \mathrm{PWS}^m[a,b]\text{: (1.1)–(1.3) hold}\right\}.$$

The above unorthodox optimal control problems have not received much attention in the related literature. In fact, it appears that (P), (P1), and (P2) have not been investigated at all. For various references pertaining to these and other multiobjective problems, fractional programming problems, and constrained optimization problems containing norms, the reader is referred to [5,6].

Recently, some applications of fractional optimal control problems have been attempted in [2–4] in the areas of finite-interval H_∞ control, performance robustness, and model reduction.

2. Preliminaries

For the most part, we shall use the same notation and terminology introduced in [6]. Here we shall recall only a few basic definitions and auxiliary results.

For $y, z \in \mathfrak{R}^\nu$, the following order notation will be used:

$y \geqq z$ if and only if $y_i \geqq z_i$ for all $i \in \underline{\nu}$;

$y \geqslant z$ if and only if $y_i \geqq z_i$ for all $i \in \underline{\nu}$, but $y \neq z$;

$y > z$ if and only if $y_i > z_i$ for all $i \in \underline{\nu}$;

$y \ngeqslant z$ is the negation of $y \geqslant z$.

Consider the multiobjective problem

$$\text{(P4)}\qquad \underset{x\in X}{\text{Minimize}}\ J(x) = \bigl(J_1(x), \ldots, J_r(x)\bigr)^T,$$

where X is a subset of a real Banach space and J_i, $i \in \underline{r}$, are real-valued functions defined on X.

An element $\bar{x}$ of X is said to be an *efficient solution* of (P4) if there is no $x \in X$ such that $J(x) \leqslant J(\bar{x})$. An $\bar{x} \in X$ is said to be a *properly efficient solution* of (P4) if it is efficient and if there exists a positive real number C such that for each $i \in \underline{r}$ and each $x \in X$ satisfying $J_i(x) < J_i(\bar{x})$, there exists at least one $j \in \underline{r}$, $j \neq i$, such that $J_j(\bar{x}) < J_j(x)$ and $[J_i(x) - J_i(\bar{x})]/[J_j(\bar{x}) - J_j(x)] \leqq C$.

In addition to this relatively more restricted form of the notion of efficiency which precludes the possibility of unbounded trade-offs between the various objectives, several other types of proper efficiency have been proposed in the literature of multiobjective programming. The relationships existing among different versions of the concept of proper efficiency have been discussed in a number of papers and books most of which are cited in [6].

With the aid of the above definitions and notation, we can now recall a set of necessary and sufficient conditions for properly efficient solutions of (P).

Theorem 2.1 [6]. *Let $(x^0, u^0) \in \mathcal{F}$, let*

$$\mu_i^0 = \frac{\int_a^b [f_i(x^0(t), u^0(t), t) + \|K_i(t)x^0(t)\|_{k(i)} + \|L_i(t)u^0(t)\|_{\ell(i)}]\,dt}{\int_a^b [g_i(x^0(t), u^0(t), t) - \|M_i(t)x^0(t)\|_{m(i)} - \|N_i(t)u^0(t)\|_{n(i)}]\,dt}, \qquad i \in \underline{r},$$

and assume that the constraints of (P) *satisfy Slater's constraint qualification* (*SCQ*); *that is, assume that there exists $(\tilde{x}, \tilde{u}) \in C^n[a,b] \times \mathrm{PWS}^m[a,b]$ such that $\tilde{x}(a) = \tilde{x}(b) = 0$ and*

$$D\tilde{x}(t) = A(t)\tilde{x}(t) + B(t)\tilde{u}(t),$$
$$h_j\bigl(\tilde{x}(t), \tilde{u}(t), t\bigr) + \bigl\|P_j(t)\tilde{x}(t)\bigr\|_{p(j)} + \bigl\|Q_j(t)\tilde{x}(t)\bigr\|_{q(j)} < 0, \qquad j \in \underline{s},$$

for all $t \in [a,b]$. Then (x^0, u^0) is a properly efficient solution of (P) *if and only if there exist $\lambda^0 \in \Lambda$, $v^0 \in \mathrm{PWS}^n[a,b]$, $w^0 \in \mathrm{PWS}_+^s[a,b]$, $\alpha^{0i} \in \mathrm{PWS}^{k_i}[a,b]$, $\beta^{0i} \in \mathrm{PWS}^{\ell_i}[a,b]$, $\gamma^{0i} \in \mathrm{PWS}^{m_i}[a,b]$, $\delta^{0i} \in \mathrm{PWS}^{n_i}[a,b]$, $i \in \underline{r}$, $\zeta^{0j} \in \mathrm{PWS}^{p_j}[a,b]$ and $\eta^{0j} \in \mathrm{PWS}^{q_j}[a,b]$, $j \in \underline{s}$, such that the following relations hold for all $t \in [a,b]$*:

$$\begin{aligned}
&\sum_{i=1}^r \lambda_i^0 \bigl\{\nabla_1 f_i\bigl(x^0(t), u^0(t), t\bigr) + K_i(t)^T \alpha^{0i}(t) \\
&\quad - \mu_i^0\bigl[\nabla_1 g_i\bigl(x^0(t), u^0(t), t\bigr) - M_i(t)^T \gamma^{0i}(t)\bigr]\bigr\} + A(t)^T v^0(t) \\
&\quad + \sum_{j=1}^s w_j^0(t)\bigl[\nabla_1 h_j\bigl(x^0(t), u^0(t), t\bigr) + P_j(t)^T \zeta^{0j}(t)\bigr] + Dv^0(t) = 0, \qquad (2.1)
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^r \lambda_i^0 \bigl\{\nabla_2 f_i\bigl(x^0(t), u^0(t), t\bigr) + L_i(t)^T \beta^{0i}(t) \\
&\quad - \mu_i^0\bigl[\nabla_2 g_i\bigl(x^0(t), u^0(t), t\bigr) - N_i(t)^T \delta^{0i}(t)\bigr]\bigr\} + B(t)^T v^0(t)
\end{aligned}$$

$$+\sum_{j=1}^{s} w_j^0(t)\big[\nabla_2 h_j\big(x^0(t), u^0(t), t\big) + Q_j(t)^T \eta^{0j}(t)\big] = 0, \tag{2.2}$$

$$\sum_{j=1}^{s} w_j^0(t)\big[h_j\big(x^0(t), u^0(t), t\big) + \big\|P_j(t)x^0(t)\big\|_{p(j)} + \big\|Q_j(t)u^0(t)\big\|_{q(j)}\big] = 0, \tag{2.3}$$

$$\big\|\alpha^{0i}(t)\big\|_{k(i)}^* \leqq 1, \quad \big\|\beta^{0i}(t)\big\|_{\ell(i)}^* \leqq 1, \quad \big\|\gamma^{0i}(t)\big\|_{m(i)}^* \leqq 1,$$
$$\big\|\delta^{0i}(t)\big\|_{n(i)}^* \leqq 1, \quad i \in \underline{r}, \tag{2.4}$$

$$\big\|\zeta^{0j}(t)\big\|_{p(j)}^* \leqq 1, \quad \big\|\eta^{0j}(t)\big\|_{q(j)}^* \leqq 1, \quad j \in \underline{s}, \tag{2.5}$$

$$\alpha^{0i}(t)^T K_i(t)x^0(t) = \big\|K_i(t)x^0(t)\big\|_{k(i)},$$
$$\beta^{0i}(t)^T L_i(t)u^0(t) = \big\|L_i(t)u^0(t)\big\|_{\ell(i)},$$
$$\gamma^{0i}(t)^T M_i(t)x^0(t) = \big\|M_i(t)x^0(t)\big\|_{m(i)},$$
$$\delta^{0i}(t)^T N_i(t)u^0(t) = \big\|N_i(t)u^0(t)\big\|_{n(i)}, \quad i \in \underline{r}, \tag{2.6}$$

$$\zeta^{0j}(t)^T P_j(t)x^0(t) = \big\|P_j(t)x^0(t)\big\|_{p(j)},$$
$$\eta^{0j}(t)^T Q_j(t)u^0(t) = \big\|Q_j(t)u^0(t)\big\|_{q(j)}, \quad j \in \underline{s}, \tag{2.7}$$

where $\Lambda = \{\lambda \in \mathfrak{R}^r\colon\ \lambda > 0,\ \sum_{i=1}^{r} \lambda_i = 1\}$, $\mathrm{PWS}_+^s[a,b] = \{w \in \mathrm{PWS}^s[a,b]\colon\ w(t) \geqq 0$ for all $t \in [a,b]\}$, $\nabla_1 f$ *and* $\nabla_2 f$ *denote the partial gradients of the function* $f : \mathfrak{R}^n \times \mathfrak{R}^m \times [a,b] \to \mathfrak{R}$, $(x(t), u(t), t) \to f(x(t), u(t), t)$ *with respect to its first and second arguments, respectively; that is,* $\nabla_1 f = (\partial f/\partial x_1(t), \ldots, \partial f/\partial x_n(t))^T$ *and* $\nabla_2 f = (\partial f/\partial u_1(t), \ldots, \partial f/\partial u_m(t))^T$, *and* $\|\cdot\|_e^*$ *denotes the dual norm to* $\|\cdot\|_e$.

The sufficiency part of the above theorem remains valid under a somewhat different set of conditions obtained by modifying (2.1) and (2.2), as shown in the following theorem. For the proof, we need the generalized Cauchy inequality [1]:

$$\text{For every } y, z \in \mathfrak{R}^N, \text{ one has } y^T z \leqq \|y\|\|z\|^*. \tag{2.8}$$

Theorem 2.2. *Let* $(x^0, u^0) \in \mathcal{F}$, *let* μ_i^0, $i \in \underline{r}$, *be as defined above, and assume that there exist* $\lambda^0, v^0, w^0, \alpha^{0i}, \beta^{0i}, \gamma^{0i}, \delta^{0i}$, $i \in \underline{r}$, ζ^{0j} *and* η^{0j}, $j \in \underline{s}$, *as specified in Theorem* 2.1, *such that* (2.3)–(2.7) *and the following inequalities hold for all* $t \in [a,b]$:

$$\begin{aligned}&\Bigg\{\sum_{i=1}^{r} \lambda_i^0 \big\{\nabla_1 f_i\big(x^0(t), u^0(t), t\big)^T + \alpha^{0i}(t)^T K_i(t) \\ &\quad - \mu_i^0\big[\nabla_1 g_i\big(x^0(t), u^0(t), t\big)^T - \gamma^{0i}(t)^T M_i(t)\big]\big\} + v^0(t)^T A(t) \\ &\quad + \sum_{j=1}^{s} w_j^0(t)\big[\nabla_1 h_j\big(x^0(t), u^0(t), t\big)^T + \zeta^{0j}(t)^T P_j(t)\big] + D v^0(t)^T\Bigg\} \\ &\quad \times \big[x(t) - x^0(t)\big] \geqq 0\end{aligned} \tag{2.9}$$

for all $x \in C^n[a,b]$ *such that* $(x,u) \in \mathcal{F}$ *for some* $u \in \mathrm{PWS}^m[a,b]$,

$$\begin{aligned}
&\Bigg\{\sum_{i=1}^{r}\lambda_i^0\big\{\nabla_2 f_i\big(x^0(t),u^0(t),t\big)^T+\beta^{0i}(t)^T L_i(t)\\
&\quad-\mu_i^0\big[\nabla_2 g_i\big(x^0(t),u^0(t),t\big)^T-\delta^{0i}(t)^T N_i(t)\big]\big\}+v^0(t)^T B(t)\\
&\quad+\sum_{j=1}^{s}w_j^0(t)\big[\nabla_2 h_j\big(x^0(t),u^0(t),t\big)^T+\eta^{0j}(t)^T Q_j(t)\big]\Bigg\}\\
&\quad\times\big[u(t)-u^0(t)\big]\geqq 0 \qquad (2.10)
\end{aligned}$$

for all $u\in \mathrm{PWS}^m[a,b]$ *such that* $(x,u)\in\mathcal{F}$ *for some* $x\in C^n[a,b]$. *Then* (x^0,u^0) *is a properly efficient solution of* (P).

Proof. Let (x,u) be an arbitrary feasible solution of (P). Then

$$\begin{aligned}
&\sum_{i=1}^{r}\lambda_i^0\int_a^b\Big\{f_i(x,u,t)+\big\|K_i(t)x\big\|_{k(i)}+\big\|L_i(t)u\big\|_{\ell(i)}\\
&\quad-\mu_i^0\big[g_i(x,u,t)-\big\|M_i(t)x\big\|_{m(i)}-\big\|N_i(t)u\big\|_{n(i)}\big]\Big\}\,dt\\
&\quad-\sum_{i=1}^{r}\lambda_i^0\int_a^b\Big\{f_i(x^0,u^0,t)+\big\|K_i(t)x^0\big\|_{k(i)}+\big\|L_i(t)u^0\big\|_{\ell(i)}\\
&\quad-\mu_i^0\big[g_i(x^0,u^0,t)-\big\|M_i(t)x^0\big\|_{m(i)}-\big\|N_i(t)u^0\big\|_{n(i)}\big]\Big\}\,dt\\
&\geqq\int_a^b\sum_{i=1}^{r}\lambda_i^0\Big\{\nabla_1 f_i(x^0,u^0,t)^T(x-x^0)+\nabla_2 f_i(x^0,u^0,t)^T(u-u^0)\\
&\quad-\mu_i^0\big[\nabla_1 g_i(x^0,u^0,t)^T(x-x^0)+\nabla_2 g_i(x^0,u^0,t)^T(u-u^0)+\big\|K_i(t)x\big\|_{k(i)}\\
&\quad+\big\|L_i(t)u\big\|_{\ell(i)}+\mu_i^0\big[\big\|M_i(t)x\big\|_{m(i)}+\big\|N_i(t)u\big\|_{n(i)}\big]-\alpha^{0i}(t)^T K_i(t)x^0\\
&\quad-\beta^{0i}(t)^T L_i(t)u^0-\mu_i^0\big[\gamma^{0i}(t)^T M_i(t)x^0+\delta^{0i}(t)^T N_i(t)u^0\big]\Big\}\,dt
\end{aligned}$$

(by the convexity of $f_i(.,.,t)$ and $-g_i(.,.,t)$, nonnegativity of λ_i^0 and μ_i^0, $i\in\underline{r}$, and (2.6))

$$\begin{aligned}
&\geqq\int_a^b\Bigg\{\sum_{i=1}^{r}\lambda_i^0\Big\{\big\|K_i(t)x\big\|_{k(i)}+\big\|L_i(t)u\big\|_{\ell(i)}-\alpha^{0i}(t)^T K_i(t)x^0-\beta^{0i}(t)^T L_i(t)u^0\\
&\quad-\alpha^{0i}(t)^T K_i(t)(x-x^0)-\beta^{0i}(t)^T L_i(t)(u-u^0)+\mu_i^0\big[\big\|M_i(t)x\big\|_{m(i)}\\
&\quad+\big\|N_i(t)u\big\|_{n(i)}-\gamma^{0i}(t)^T M_i(t)x^0-\delta^{0i}(t)^T N_i(t)u^0-\gamma^{0i}(t)^T M_i(t)(x-x^0)\\
&\quad-\delta^{0i}(t)^T N_i(t)(u-u^0)\big]\Big\}-\Bigg\{v^0(t)^T A(t)+\sum_{j=1}^{s}w_j^0(t)\big[\nabla_1 h_j(x^0,u^0,t)^T
\end{aligned}$$

$$+ \zeta^{0j}(t)^T P_j(t)\big] + Dv^0(t)^T \Big\}(x - x^0) - \Big\{ v^0(t)^T B(t)$$

$$+ \sum_{j=1}^{s} w_j^0(t)\big[\nabla_2 h_j(x^0, u^0, t)^T + \eta^{0j}(t)^T Q_j(t)\big]\Big\}(u - u^0)\Big\}\, dt$$

(by (2.9) and (2.10))

$$\geqq \int_a^b \Big\{ \sum_{i=1}^{r} \lambda_i^0 \Big\{ \|K_i(t)x\|_{k(i)} + \|L_i(t)u\|_{\ell(i)} - \|K_i(t)x\|_{k(i)} \|\alpha^{0i}(t)\|_{k(i)}^*$$

$$- \|L_i(t)u\|_{\ell(i)} \|\beta^{0i}(t)\|_{\ell(i)}^* + \mu_i^0\big[\|M_i(t)x\|_{m(i)} + \|N_i(t)u\|_{n(i)}$$

$$- \|M_i(t)x\|_{m(i)} \|\gamma^{0i}(t)\|_{m(i)}^* - \|N_i(t)u\|_{n(i)} \|\delta^{0i}(t)\|_{n(i)}^* \Big\}$$

$$+ v^0(t)^T\big[D(x - x^0) - A(t)(x - x^0) - B(t)(u - u^0)\big]$$

$$- \sum_{j=1}^{s} w_j^0(t)\Big[\nabla_1 h_j(x^0, u^0, t)^T (x - x^0) + \nabla_2 h_j(x^0, u^0, t)^T (u - u^0)$$

$$+ \|P_j(t)x\|_{p(j)} \|\zeta^{0j}(t)\|_{p(j)}^* - \zeta^{0j}(t)^T P_j(t)x^0$$

$$+ \|Q_j(t)u\|_{q(j)} \|\eta^{0j}(t)\|_{q(j)}^* - \eta^{0j}(t)^T Q_j(t)u^0\Big]\Big\}\, dt$$

(by the nonnegativity of λ_i^0, μ_i^0, $i \in \underline{r}$, and $w^0(t)$, integration by parts, and (2.8))

$$\geqq \int_a^b \Big\{ v^0(t)^T\big[Dx - A(t)x - B(t)u\big] - v^0(t)^T\big[Dx^0 - A(t)x^0 - B(t)u^0\big]$$

$$+ \sum_{j=1}^{s} w_j^0(t)\big[h_j(x^0, u^0, t) + \zeta^{0j}(t)^T P_j(t)x^0 + \eta^{0j}(t)^T Q_j(t)u^0\big]$$

$$- \sum_{j=1}^{s} w_j^0(t)\big[h_j(x, u, t) + \|P_j(t)x\|_{p(j)} + \|Q_j(t)u\|_{q(j)}\big]\Big\}\, dt$$

(by the nonnegativity of λ_i^0, μ_i^0, $i \in \underline{r}$, and $w^0(t)$, convexity of $h_j(.,.,t)$, $j \in \underline{s}$, (2.4), and (2.5))

$\geqq 0$ (by the feasibility of (x, u) and (x^0, u^0), nonnegativity of $w^0(t)$, (2.3), and (2.7)).

Since $(x, u) \in \mathcal{F}$ was arbitrary, this inequality implies that (x^0, u^0) is an optimal solution of the single-objective problem

$$\underset{(x,u)\in\mathcal{F}}{\text{Minimize}} \sum_{i=1}^{r} \lambda_i^0 \int_a^b \Big\{ f_i(x,u,t) + \big\|K_i(t)x\big\|_{k(i)} + \big\|L_i(t)u\big\|_{\ell(i)}$$

$$- \mu_i^0\big[g_i(x,u,t) - \big\|M_i(t)x\big\|_{m(i)} - \big\|N_i(t)u\big\|_{n(i)}\big]\Big\}\,dt.$$

Now it follows from Lemmas 2.1 and 3.1 of [6] that (x^0, u^0) is a properly efficient solution of (P). □

In the above proof, the argument t of the functions x, x^0, u, and u^0 was omitted for the sake of notational simplicity. This practice will be continued throughout the sequel.

3. Semiparametric proper efficiency conditions

The proper efficiency conditions stated in Theorems 2.1 and 2.2 depend on the parameters λ_i^0 and μ_i^0, $i \in \underline{r}$, which were introduced as a result of an indirect approach employed in [6] requiring two auxiliary parametric problems. In this section we shall eliminate one of these two sets of parameters and, consequently, obtain some semiparametric necessary and sufficient conditions for properly efficient solutions of (P).

Theorem 3.1. *Let* $(x^*, u^*) \in \mathcal{F}$ *and assume that the constraints of* (P) *satisfy SCQ* (*see Theorem* 2.1). *Then* (x^*, u^*) *is a properly efficient solution of* (P) *if and only if there exist* $\lambda^* \in \Lambda$, $v^* \in \mathrm{PWS}^n[a,b]$, $w^* \in \mathrm{PWS}_+^s[a,b]$, $\alpha^{*i} \in \mathrm{PWS}^{k_i}[a,b]$, $\beta^{*i} \in \mathrm{PWS}^{\ell_i}[a,b]$, $\gamma^{*i} \in \mathrm{PWS}^{m_i}[a,b]$, $\delta^{*i} \in \mathrm{PWS}^{n_i}[a,b]$, $i \in \underline{r}$, $\zeta^{*j} \in \mathrm{PWS}^{p_j}[a,b]$ *and* $\eta^{*j} \in \mathrm{PWS}^{q_j}[a,b]$, $j \in \underline{s}$, *such that the following relations hold for all* $t \in [a,b]$:

$$\sum_{i=1}^{r} \lambda_i^* \Big\{ \Gamma_i(x^*,u^*)\big[\nabla_1 f_i(x^*,u^*,t) + K_i(t)^T \alpha^{*i}(t)\big]$$

$$- \Phi_i(x^*,u^*)\big[\nabla_1 g_i(x^*,u^*,t) - M_i(t)^T \gamma^{*i}(t)\big]\Big\} + A(t)^T v^*(t)$$

$$+ \sum_{j=1}^{s} w_j^*(t)\big[\nabla_1 h_j(x^*,u^*,t) + P_j(t)^T \zeta^{*j}(t)\big] + Dv^*(t) = 0, \tag{3.1}$$

$$\sum_{i=1}^{r} \lambda_i^* \Big\{ \Gamma_i(x^*,u^*)\big[\nabla_2 f_i(x^*,u^*,t) + L_i(t)^T \beta^{*i}(t)\big]$$

$$- \Phi_i(x^*,u^*)\big[\nabla_2 g_i(x^*,u^*,t) - N_i(t)^T \delta^{*i}(t)\big]\Big\} + B(t)^T v^*(t)$$

$$+ \sum_{j=1}^{s} w_j^*(t)\big[\nabla_2 h_j(x^*,u^*,t) + Q_j(t)^T \eta^{*j}(t)\big] = 0, \tag{3.2}$$

$$\sum_{j=1}^{s} w_j^*(t)\big[h_j(x^*,u^*,t) + \big\|P_j(t)x^*\big\|_{p(j)} + \big\|Q_j(t)u^*\big\|_{q(j)}\big] = 0, \tag{3.3}$$

$$\|\alpha^{*i}(t)\|_{k(i)}^{*} \leqq 1, \quad \|\beta^{*i}(t)\|_{\ell(i)}^{*} \leqq 1, \quad \|\gamma^{*i}(t)\|_{m(i)}^{*} \leqq 1,$$
$$\|\delta^{*i}(t)\|_{n(i)}^{*} \leqq 1, \quad i \in \underline{r}, \tag{3.4}$$
$$\|\zeta^{*j}(t)\|_{p(j)}^{*} \leqq 1, \quad \|\eta^{*j}(t)\|_{q(j)}^{*} \leqq 1, \quad j \in \underline{s}, \tag{3.5}$$
$$\alpha^{*i}(t)^T K_i(t)x^* = \|K_i(t)x^*\|_{k(i)}, \quad \beta^{*i}(t)^T L_i(t)u^* = \|L_i(t)u^*\|_{\ell(i)},$$
$$\gamma^{*i}(t)^T M_i(t)x^* = \|M_i(t)x^*\|_{m(i)}, \quad \delta^{*i}(t)^T N_i(t)u^* = \|N_i(t)u^*\|_{n(i)},$$
$$i \in \underline{r}, \tag{3.6}$$
$$\zeta^{*j}(t)^T P_j(t)x^* = \|P_j(t)x^*\|_{p(j)}, \quad \eta^{*j}(t)^T Q_j(t)u^* = \|Q_j(t)u^*\|_{q(j)},$$
$$j \in \underline{s}, \tag{3.7}$$

where

$$\Phi_i(x^*, u^*) = \int_a^b \left[f_i(x^*, u^*, t) + \|K_i(t)x^*\|_{k(i)} + \|L_i(t)u^*\|_{\ell(i)} \right] dt$$

and

$$\Gamma_i(x^*, u^*) = \int_a^b \left[g_i(x^*, u^*, t) - \|M_i(t)x^*\|_{m(i)} - \|N_i(t)u^*\|_{n(i)} \right] dt, \quad i \in \underline{r}.$$

Proof. By Theorem 2.1, there exist $\lambda^0 \in \Lambda$, $v^0 \in \mathrm{PWS}^n[a, b]$, $w^0 \in \mathrm{PWS}_+^s[a, b]$, and α^{*i}, β^{*i}, γ^{*i}, δ^{*i}, $i \in \underline{r}$, ζ^{*j} and η^{*j}, $j \in \underline{s}$, as specified above, such that (3.4)–(3.7) and the following relations hold for all $t \in [a, b]$:

$$\sum_{i=1}^{r} \lambda_i^0 \Big\{ \nabla_1 f_i(x^*, u^*, t) + K_i(t)^T \alpha^{*i}(t) - \left[\Phi_i(x^*, u^*)/\Gamma_i(x^*, u^*)\right]$$
$$\times \left[\nabla_1 g_i(x^*, u^*, t) - M_i(t)^T \gamma^{*i}(t)\right] \Big\} + A(t)^T v^0(t)$$
$$+ \sum_{j=1}^{s} w_j^0(t) \left[\nabla_1 h_j(x^*, u^*, t) + P_j(t)^T \zeta^{*j}(t)\right] + D v^0(t) = 0,$$

$$\sum_{i=1}^{r} \lambda_i^0 \Big\{ \nabla_2 f_i(x^*, u^*, t) + L_i(t)^T \beta^{*i}(t) - \left[\Phi_i(x^*, u^*)/\Gamma_i(x^*, u^*)\right]$$
$$\times \left[\nabla_2 g_i(x^*, u^*, t) - N_i(t)^T \delta^{*i}(t)\right] \Big\} + B(t)^T v^0(t)$$
$$+ \sum_{j=1}^{s} w_j^0(t) \left[\nabla_2 h_j(x^*, u^*, t) + Q_j(t)^T \eta^{*j}(t)\right] = 0,$$

$$\sum_{j=1}^{s} w_j^0(t) \left[h_j(x^*, u^*, t) + \|P_j(t)x^*\|_{p(j)} + \|Q_j(t)u^*\|_{q(j)} \right] = 0.$$

Since $\lambda_i^0 > 0$, $\Gamma_i(x^*, u^*) > 0$ for all $i \in \underline{r}$, and hence $c \equiv \sum_{i=1}^r [\lambda_i^0/\Gamma_i(x^*, u^*)] > 0$, the last three equations can be rewritten as follows:

$$\begin{aligned}
&\sum_{i=1}^r \big[\lambda_i^0/c\Gamma_i(x^*, u^*)\big]\Big\{\Gamma_i(x^*, u^*)\big[\nabla_1 f_i(x^*, u^*, t) + K_i(t)^T\alpha^{*i}(t)\big] \\
&\quad - \Phi_i(x^*, u^*)\big[\nabla_1 g_i(x^*, u^*, t) + M_i(t)^T\gamma^{*i}(t)\big]\Big\} + A(t)^T\big[v^0(t)/c\big] \\
&\quad + \sum_{j=1}^s \big[w_j^0(t)/c\big]\big[\nabla_1 h_j(x^*, u^*, t) + P_j(t)^T\zeta^{*j}(t)\big] + D\big[v^0(t)/c\big] = 0,
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
&\sum_{i=1}^r \big[\lambda_i^0/c\Gamma_i(x^*, u^*)\big]\Big\{\Gamma_i(x^*, u^*)\big[\nabla_2 f_i(x^*, u^*, t) + L_i(t)^T\beta^{*i}(t)\big] \\
&\quad - \Phi_i(x^*, u^*)\big[\nabla_2 g_i(x^*, u^*, t) - N_i(t)^T\delta^{*i}(t)\big]\Big\} + B(t)^T\big[v^0(t)/c\big] \\
&\quad + \sum_{j=1}^s \big[w_j^0(t)/c\big]\big[\nabla_2 h_j(x^*, u^*, t) + Q_j(t)^T\zeta^{*j}(t)\big] = 0,
\end{aligned} \tag{3.9}$$

$$\sum_{j=1}^s \big[w_j^0(t)/c\big]\big[h_j(x^*, u^*, t) + \|P_j(t)x^*\|_{p(j)} + \|Q_j(t)u^*\|_{q(j)}\big] = 0. \tag{3.10}$$

Now letting $\lambda_i^* = \lambda_i^0/c\Gamma_i(x^*, u^*)$, $i \in \underline{r}$, $v^* = v^0/c$, and $w^* = w^0/c$ in (3.8)–(3.10), we see that (3.1)–(3.3) also hold. Since by reversing the above process one can always transform (3.2) and (3.3) into (2.1) and (2.2), respectively, the sufficiency assertion of the theorem follows from Theorem 2.1. □

The next theorem can be proved by using the reverse of the process employed in the proof of the necessity part of Theorem 3.1, and appealing to Theorem 2.2.

Theorem 3.2. *Let $(x^*, u^*) \in \mathcal{F}$ and assume that there exist λ^*, v^*, w^*, α^{*i}, β^{*i} ,γ^{*i}, δ^{*i}, $i \in \underline{r}$, ζ^{*j} and η^{*j}, $j \in \underline{s}$, as specified in Theorem* 3.1, *such that* (3.3)–(3.7) *and the following inequalities hold for all $t \in [a, b]$:*

$$\begin{aligned}
&\Bigg\{\sum_{i=1}^r \lambda_i^*\Big\{\Gamma_i(x^*, u^*)\big[\nabla_1 f_i(x^*, u^*, t)^T + \alpha^{*i}(t)^T K_i(t)\big] \\
&\quad - \Phi_i(x^*, u^*)\big[\nabla_1 g_i(x^*, u^*, t)^T - \gamma^{*i}(t)^T M_i(t)\big]\Big\} + v^*(t)^T A(t) \\
&\quad + \sum_{j=1}^s w_j^*(t)\big[\nabla_1 h_j(x^*, u^*, t)^T + \zeta^{*j}(t)^T P_j(t)\big] + Dv^*(t)^T\Bigg\}(x - x^*) \geqq 0,
\end{aligned}$$

for all $x \in C^n[a, b]$ such that $(x, u) \in \mathcal{F}$ for some $u \in \mathrm{PWS}^m[a, b]$,

$$\Bigg\{\sum_{i=1}^r \lambda_i^*\Big\{\Gamma_i(x^*, u^*)\big[\nabla_2 f_i(x^*, u^*, t)^T + \beta^{*i}(t)^T L_i(t)\big]$$

$$- \Phi_i(x^*, u^*)\big[\nabla_2 g_i(x^*, u^*, t)^T - \delta^{*i}(t)^T N_i(t)\big]\Big\} + v^*(t)^T B(t)$$
$$+ \sum_{j=1}^{s} w_j^*(t)\big[\nabla_2 h_j(x^*, u^*, t)^T + \eta^{*j}(t)^T Q_j(t)\big]\Bigg\}(u - u^*) \geqq 0$$

for all $u \in \mathrm{PWS}^m[a, b]$ *such that* $(x, u) \in \mathcal{F}$ *for some* $x \in C^n[a, b]$. *Then* (x^*, u^*) *is a properly efficient solution of* (P).

The two sets of proper efficiency conditions stated above will lead to the formulation of two types of duality models for (P). We shall briefly elaborate on the differences between these models in the following section.

4. Duality model I

Making use of the form and contents of the proper efficiency criteria presented in the preceding section, we shall next formulate four semiparametric duality models for (P) and prove appropriate duality theorems.

Consider the following two problems:

(DI) Maximize $\big(\Phi_1(x, u)/\Gamma_1(x, u), \ldots, \Phi_r(x, u)/\Gamma_r(x, u)\big)^T$

subject to

$$x(a) = 0, \qquad x(b) = 0, \tag{4.1}$$

$$\sum_{i=1}^{r} \lambda_i \Big\{\Gamma_i(x, u)\big[\nabla_1 f_i(x, u, t) + K_i(t)^T \alpha^i(t)\big] - \Phi_i(x, u)\big[\nabla_1 g_i(x, u, t) - M_i(t)^T \gamma^i(t)\big]\Big\} + A(t)^T v(t) + \sum_{j=1}^{s} w_j(t)\big[\nabla_1 h_j(x, u, t) + P_j(t)^T \zeta^j(t)\big] + Dv(t) = 0, \quad t \in [a, b], \tag{4.2}$$

$$\sum_{i=1}^{r} \lambda_i \Big\{\Gamma_i(x, u)\big[\nabla_2 f_i(x, u, t) + L_i(t)^T \beta^i(t)\big] - \Phi_i(x, u)\big[\nabla_2 g_i(x, u, t) - N_i(t)^T \delta^i(t)\big]\Big\} + B(t)^T v(t) + \sum_{j=1}^{s} w_j(t)\big[\nabla_2 h_j(x, u, t) + Q_j(t)^T \eta^j(t)\big] = 0, \quad t \in [a, b], \tag{4.3}$$

$$v(t)^T\big[-Dx + A(t)x + B(t)u\big] + \sum_{j=1}^{s} w_j(t)\big[h_j(x, u, t) + \big\|P_j(t)x\big\|_{p(j)} + \big\|Q_j(t)u\big\|_{q(j)}\big] \geqq 0, \quad t \in [a, b], \tag{4.4}$$

$$\big\|\alpha^i(t)\big\|^*_{k(i)} \leqq 1, \quad \big\|\beta^i(t)\big\|^*_{\ell(i)} \leqq 1, \quad \big\|\gamma^i(t)\big\|^*_{m(i)} \leqq 1,$$

$$\big\|\delta^i(t)\big\|^*_{n(i)} \leqq 1, \quad t \in [a,b],\ i \in \underline{r}, \tag{4.5}$$

$$\big\|\zeta^j(t)\big\|^*_{p(j)} \leqq 1, \quad \big\|\eta^j(t)\big\|^*_{q(j)} \leqq 1, \quad t \in [a,b],\ j \in \underline{s}, \tag{4.6}$$

$$\alpha^i(t)^T K_i(t)x = \big\|K_i(t)x\big\|_{k(i)}, \quad \beta^i(t)^T L_i(t)u = \big\|L_i(t)u\big\|_{\ell(i)},$$
$$\gamma^i(t)^T M_i(t)x = \big\|M_i(t)x\big\|_{m(i)}, \quad \delta^i(t)^T N_i(t)u = \big\|N_i(t)u\big\|_{n(i)},$$
$$t \in [a,b],\ i \in \underline{r}, \tag{4.7}$$

$$\zeta^j(t)^T P_j(t)x = \big\|P_j(t)x\big\|_{p(j)}, \quad \eta^j(t)^T Q_j(t)u = \big\|Q_j(t)u\big\|_{q(j)},$$
$$t \in [a,b],\ j \in \underline{s}, \tag{4.8}$$

$$x \in C^n[a,b], \quad u \in \mathrm{PWS}^m[a,b], \quad \lambda \in \Lambda, \quad v \in \mathrm{PWS}^n[a,b],$$
$$w \in \mathrm{PWS}^s_+[a,b], \quad \alpha^i \in \mathrm{PWS}^{k_i}[a,b], \quad \beta^i \in \mathrm{PWS}^{\ell_i}[a,b],$$
$$\gamma^i \in \mathrm{PWS}^{m_i}[a,b], \quad \delta^i \in \mathrm{PWS}^{n_i}[a,b], \quad i \in \underline{r},$$
$$\zeta^j \in \mathrm{PWS}^{p_j}[a,b], \quad \eta^j \in \mathrm{PWS}^{q_j}[a,b], \quad j \in \underline{s}; \tag{4.9}$$

($\widetilde{\mathrm{D}}$I) Maximize $\big(\Phi_1(x,u)/\Gamma_1(x,u), \ldots, \Phi_r(x,u)/\Gamma_r(x,u)\big)^T$

subject to (4.1), (4.4)–(4.9), and

$$\Bigg\{\sum_{i=1}^r \lambda_i \Big\{\Gamma_i(x,u)\big[\nabla_1 f_i(x,u,t)^T + \alpha^i(t)^T K_i(t)\big]$$
$$- \Phi_i(x,u)\big[\nabla_1 g_i(x,u,t)^T - \gamma^i(t)^T M_i(t)\big]\Big\} + v(t)^T A(t)$$
$$+ \sum_{j=1}^s w_j(t)\big[\nabla_1 h_j(x,u,t)^T + \zeta^j(t)^T P_j(t)\big] + Dv(t)^T\Bigg\}(\bar{x} - x) \geqq 0$$

for all $t \in [a,b]$ and all $\bar{x} \in C^n[a,b]$ such that $(\bar{x}, u) \in \mathcal{F}$ for some $u \in \mathrm{PWS}^m[a,b]$, (4.10)

$$\Bigg\{\sum_{i=1}^r \lambda_i \Big\{\Gamma_i(x,u)\big[\nabla_2 f_i(x,u,t)^T + \beta^i(t)^T L_i(t)\big]$$
$$- \Phi_i(x,u)\big[\nabla_2 g_i(x,u,t)^T - \delta^i(t)^T N_i(t)\big]\Big\} + v(t)^T B(t)$$
$$+ \sum_{j=1}^s w_j(t)\big[\nabla_2 h_j(x,u,t)^T + \eta^j(t)^T Q_j(t)\big]\Bigg\}(\bar{u} - u) \geqq 0$$

for all $t \in [a,b]$ and all $\bar{u} \in \mathrm{PWS}^m[a,b]$ such that $(x, \bar{u}) \in \mathcal{F}$ for some $x \in C^n[a,b]$. (4.11)

Comparing (DI) and ($\widetilde{\mathrm{D}}$I), we observe that ($\widetilde{\mathrm{D}}$I) is relatively more general than (DI) in the sense that any feasible solution of (DI) is also feasible for ($\widetilde{\mathrm{D}}$I), but the converse is not necessarily true. Moreover, we see that (4.2) and (4.3) together form a system of $n + m$ equations, whereas (4.10) and (4.11) are two inequalities which in general cannot be transformed to equivalent systems of equations. Therefore, (DI) and ($\widetilde{\mathrm{D}}$I) are essentially different dual problems and, depending on the properties of the primal problem under consideration, one of these dual problems may be preferable to the other.

Despite these apparent differences, however, it turns out that the statements and proofs of the duality theorems for (P)–(DI) and (P)–($\widetilde{\mathrm{D}}$I) are almost identical and, therefore, we shall state and prove these theorems only for the pair (P)–(DI).

Throughout this section and the next, it will be assumed that $\Phi_i(x,u) \geqq 0$ and $\Gamma_i(x,u) > 0$ for all $i \in \underline{r}$ and all (x,u) such that $(x,u,\lambda,v,w,\alpha^1,\dots,\alpha^r,\beta^1,\dots,\beta^r,\gamma^1,\dots,\gamma^r,\delta^1,\dots,\delta^r,\zeta^1,\dots,\zeta^s,\eta^1,\dots,\eta^s)$ is a feasible solution of the dual problem under consideration.

The next two theorems show that (DI) is a dual problem for (P).

Theorem 4.1 (Weak duality). *Let $(\bar{x},\bar{u})$ and $(x,z) \equiv (x,u,\lambda,v,w,\alpha^1,\dots,\alpha^r,\beta^1,\dots,\beta^r,\gamma^1,\dots,\gamma^r,\delta^1,\dots,\delta^r,\zeta^1,\dots,\zeta^s,\eta^1,\dots,\eta^s)$ be arbitrary feasible solutions of* (P) *and* (DI)*, respectively. Then $\theta(\bar{x},\bar{u}) \nleq \varphi_{\mathrm{I}}(x,z)$, where $\theta = (\theta_1\dots,\theta_r)^T$ and $\varphi_{\mathrm{I}} = (\varphi_{\mathrm{I}1},\dots,\varphi_{\mathrm{I}r})^T$ are the objective functions of* (P) *and* (DI)*, respectively.*

Proof. Since

$$\begin{aligned}
&\sum_{i=1}^{r}\lambda_i\big[\Gamma_i(x,u)\Phi_i(\bar{x},\bar{u}) - \Phi_i(x,u)\Gamma_i(\bar{x},\bar{u})\big] \\
&\quad= \sum_{i=1}^{r}\lambda_i\Bigg\{\int_a^b\big[g_i(x,u,t) - \|M_i(t)x\|_{m(i)} - \|N_i(t)u\|_{n(i)}\big]\,dt \\
&\qquad\qquad \times \int_a^b\big[f_i(\bar{x},\bar{u},t) + \|K_i(t)\bar{x}\|_{k(i)} + \|L_i(t)\bar{u}\|_{\ell(i)}\big]\,dt \\
&\qquad - \int_a^b\big[f_i(x,u,t) + \|K_i(t)x\|_{k(i)} + \|L_i(t)u\|_{\ell(i)}\big]\,dt \\
&\qquad\qquad \times \int_a^b\big[g_i(\bar{x},\bar{u},t) - \|M_i(t)\bar{x}\|_{m(i)} - \|N_i(t)\bar{u}\|_{n(i)}\big]\,dt\Bigg\} \\
&\quad= \sum_{i=1}^{r}\lambda_i\int_a^b\Big\{\Gamma_i(x,u)\big[f_i(\bar{x},\bar{u},t) - f_i(x,u,t)\big] \\
&\qquad - \Phi_i(x,u)\big[g_i(\bar{x},\bar{u},t) - g_i(x,u,t)\big] + \Gamma_i(x,u)\big[\|K_i(t)\bar{x}\|_{k(i)} + \|L_i(t)\bar{u}\|_{\ell(i)} \\
&\qquad - \|K_i(t)x\|_{k(i)} - \|L_i(t)u\|_{\ell(i)}\big] + \Phi_i(x,u)\big[\|M_i(t)\bar{x}\|_{m(i)} + \|N_i(t)\bar{u}\|_{n(i)} \\
&\qquad - \|M_i(t)x\|_{m(i)} - \|N_i(t)u\|_{n(i)}\big]\Big\}\,dt
\end{aligned}$$

$$\geqq \sum_{i=1}^{r} \lambda_i \int_a^b \Big\{ \Gamma_i(x,u)\big\{\nabla_1 f_i(x,u,t)^T(\bar{x}-x) + \nabla_2 f_i(x,u,t)^T(\bar{u}-u)\big\}$$
$$- \Phi_i(x,u)\big\{\nabla_1 g_i(x,u,t)^T(\bar{x}-x) + \nabla_2 g_i(x,u,t)^T(\bar{u}-u)\big\}$$
$$+ \Gamma_i(x,u)\big[\|K_i(t)\bar{x}\|_{k(i)} + \|L_i(t)\bar{u}\|_{\ell(i)} - \alpha^i(t)^T K_i(t)x$$
$$- \beta^i(t)^T L_i(t)u\big] + \Phi_i(x,u)\big[\|M_i(t)\bar{x}\|_{m(i)} + \|N_i(t)\bar{u}\|_{n(i)}$$
$$- \gamma^i(t)^T M_i(t)x - \delta^i(t)^T N_i(t)u\big]\Big\}\,dt$$

(by the convexity of $f_i(.,.,t)$ and $-gj(.,.,t)$, nonnegativity of λ_i, $\Phi_i(x,u)$, and $\Gamma_i(x,u)$, $i \in \underline{r}$, and (4.7))

$$= \int_a^b \Bigg\{ \sum_{i=1}^{r} \lambda_i \Big\{ \Gamma_i(x,u)\big[\|K_i(t)\bar{x}\|_{k(i)} + \|L_i(t)\bar{u}\|_{\ell(i)} - \alpha^i(t)^T K_i(t)x$$
$$- \beta^i(t)^T L_i(t)u - \alpha^i(t)^T K_i(t)(\bar{x}-x) - \beta^i(t)^T L_i(t)(\bar{u}-u)\big]$$
$$+ \Phi_i(x,u)\big[\|M_i(t)\bar{x}\|_{m(i)} + \|N_i(t)\bar{u}\|_{n(i)} - \gamma^i(t)^T M_i(t)x - \delta^i(t)^T N_i(t)u$$
$$- \gamma^i(t)^T M_i(t)(\bar{x}-x) - \delta^i(t)^T N_i(t)(\bar{u}-u)\big]\Big\}$$
$$- \Bigg\{ v(t)^T A(t) + \sum_{j=1}^{s} w_j(t)\big[\nabla_1 h_j(x,u,t)^T + \zeta^j(t)^T P_j(t)\big] + Dv(t)^T \Bigg\}(\bar{x}-x)$$
$$- \Bigg\{ v(t)^T B(t) + \sum_{j=1}^{s} w_j(t)\big[\nabla_2 h_j(x,u,t)^T + \eta^j(t)^T Q_j(t)\big]\Bigg\}(\bar{u}-u)\Bigg\}\,dt$$

(by (4.2) and (4.3))

$$\geqq \int_a^b \Bigg\{ \sum_{i=1}^{r} \lambda_i \Big\{ \Gamma_i(x,u)\big[\|K_i(t)\bar{x}\|_{k(i)} + \|L_i(t)\bar{u}\|_{\ell(i)} - \|K_i(t)\bar{x}\|_{k(i)}\|\alpha^i(t)\|^*_{k(i)}$$
$$- \|L_i(t)\bar{u}\|_{\ell(i)}\|\beta^i(t)\|^*_{\ell(i)}\big] + \Phi_i(x,u)\big[\|M_i(t)\bar{x}\|_{m(i)} + \|N_i(t)\bar{u}\|_{n(i)}$$
$$- \|M_i(t)\bar{x}\|_{m(i)}\|\gamma^i(t)\|^*_{m(i)} - \|N_i(t)\bar{u}\|_{n(i)}\|\delta^i(t)\|^*_{n(i)}\big]\Big\}$$
$$+ v(t)^T\big[D(\bar{x}-x) - A(t)(\bar{x}-x) - B(t)(\bar{u}-u)\big]$$
$$- \sum_{j=1}^{s} w_j(t)\big[\nabla_1 h_j(x,u,t)^T(\bar{x}-x) + \nabla_2 h_j(x,u,t)^T(\bar{u}-u)$$
$$+ \|P_j(t)\bar{x}\|_{p(j)}\|\zeta^j(t)\|^*_{p(j)} - \zeta^j(t)^T P_j(t)x$$
$$+ \|Q_j(t)\bar{u}\|_{q(j)}\|\eta^j(t)\|^*_{q(j)} - \eta^j(t)^T Q_j(t)u\big]\Bigg\}\,dt$$

(by the nonnegativity of λ_i, $\Phi_i(x,u)$, $\Gamma_i(x,u)$, $i \in \underline{r}$, and $w(t)$, integration by parts, and (2.8))

$$\geqq \int_a^b \Bigg\{ v(t)^T \big[D\bar{x} - A(t)\bar{x} - B(t)\bar{u}\big] - v(t)^T \big[Dx - A(t)x - B(t)u\big]$$

$$+ \sum_{j=1}^s w_j(t)\big[h_j(x,u,t) + \zeta^j(t)^T P_j(t)x + \eta^j(t)^T Q_j(t)u\big]$$

$$- \sum_{j=1}^s w_j(t)\big[h_j(\bar{x},\bar{u},t) + \big\|P_j(t)\bar{x}\big\|_{p(j)} + \big\|Q_j(t)\bar{u}\big\|_{q(j)}\big] \Bigg\}\, dt$$

(by the nonnegativity of λ_i, $\Phi_i(x,u)$, $\Gamma_i(x,u)$, $i \in \underline{r}$, and $w(t)$, convexity of $h_j(.,.,t)$, $j \in \underline{s}$, (4.5), and (4.6))

$$\geqq 0 \quad \text{(by the primal feasibility of } (\bar{x},\bar{u}),\ \text{nonnegativity of } w(t),\ (4.4),\ \text{and } (4.8)), \tag{4.12}$$

it follows that

$$\big(\Gamma_1(x,u)\Phi_1(\bar{x},\bar{u}) - \Phi_1(x,u)\Gamma_1(\bar{x},\bar{u}), \ldots, \Gamma_r(x,u)\Phi_r(\bar{x},\bar{u}) - \Phi_r(x,u)\Gamma_r(\bar{x},\bar{u})\big)^T \nleq (0,\ldots,0)^T,$$

which implies that

$$\begin{aligned}\theta(\bar{x},\bar{u}) &= \big(\Phi_1(\bar{x},\bar{u})/\Gamma_1(\bar{x},\bar{u}), \ldots, \Phi_r(\bar{x},\bar{u})/\Gamma_r(\bar{x},\bar{u})\big)^T \\ &\nleq \big(\Phi_1(x,u)/\Gamma_1(x,u), \ldots, \Phi_r(x,u)/\Gamma_r(x,u)\big)^T \\ &= \varphi_{\mathrm{I}}(x,z). \qquad \Box\end{aligned}$$

Theorem 4.2 (Strong duality). *Let (x^*,u^*) be a properly efficient solution of* (P) *and assume that the constraints of* (P) *satisfy SCQ. Then there exist $\lambda^* \in \Lambda$, $v^* \in \mathrm{PWS}^n[a,b]$, $w^* \in PWS^s_+[a,b]$, $\alpha^{*i} \in \mathrm{PWS}^{k_i}[a,b]$, $\beta^{*i} \in \mathrm{PWS}^{\ell_i}[a,b]$, $\gamma^{*i} \in \mathrm{PWS}^{m_i}[a,b]$, $\delta^{*i} \in \mathrm{PWS}^{n_i}[a,b]$, $i \in \underline{r}$, $\zeta^{*j} \in \mathrm{PWS}^{p_j}[a,b]$ and $\eta^{*j} \in \mathrm{PWS}^{q_j}[a,b]$, $j \in \underline{s}$, such that $(x^*,z^*) = (x^*,u^*,\lambda^*,v^*,w^*,\alpha^{*1},\ldots,\alpha^{*r},\beta^{*1},\ldots,\beta^{*r},\gamma^{*1},\ldots,\gamma^{*r},\delta^{*1},\ldots,\delta^{*r},\zeta^{*1},\ldots,\zeta^{*s},\eta^{*1},\ldots,\eta^{*s})$ is a properly efficient solution of* (DI) (*and $\theta(x^*,u^*) = \varphi_{\mathrm{I}}(x^*,z^*)$*).

Proof. By Theorem 3.1, there exist $\lambda^*, v^*, w^*, \alpha^{*i}, \beta^{*i}, \gamma^{*i}, \delta^{*i}$, $i \in \underline{r}$, ζ^{*j} and η^{*j}, $j \in \underline{s}$, as specified above, such that (x^*,z^*) is a feasible solution of (DI). If (x^*,z^*) were not efficient, then there would exist a feasible solution $(\bar{x},\bar{z}) = (\bar{x},\bar{u},\bar{\lambda},\bar{v},\bar{w},\bar{\alpha}^1,\ldots,\bar{\alpha}^r,\bar{\beta}^1,\ldots,\bar{\beta}^r,\bar{\gamma}^1,\ldots,\bar{\gamma}^r,\bar{\delta}^1,\ldots,\bar{\delta}^r,\bar{\zeta}^1,\ldots,\bar{\zeta}^s,\bar{\eta}^1,\ldots,\bar{\eta}^s)$ of (DI) such that $\varphi_{\mathrm{I}i}(\bar{x},\bar{z}) \geqq \varphi_{\mathrm{I}i}(x^*,z^*)$ for all $i \in \underline{r}$, and $\varphi_{\mathrm{I}j}(\bar{x},\bar{z}) > \varphi_{\mathrm{I}j}(x^*,z^*)$ for at least one $j \in \underline{r}$. Since $\bar{\lambda} > 0$, these inequalities imply that

$$\sum_{i=1}^r \bar{\lambda}_i \big[\Gamma_i(\bar{x},\bar{u})\Phi_i(x^*,u^*) - \Phi_i(\bar{x},\bar{u})\Gamma_i(x^*,u^*)\big] < 0$$

which contradicts (4.12) (with $(\bar{x},\bar{u})$ replaced by (x^*,u^*) and (x,z) by $(\bar{x},\bar{z})$). Therefore, (x^*,z^*) is an efficient solution of (DI). It remains to show that it is properly efficient. Suppose to the contrary that it is not. Then there exists a feasible solution $(\tilde{x},\tilde{z}) =$

$(\tilde{x}, \tilde{u}, \tilde{\lambda}, \tilde{v}, \tilde{w}, \tilde{\alpha}^1, \ldots, \tilde{\alpha}^r, \tilde{\beta}^1, \ldots, \tilde{\beta}^r, \tilde{\gamma}^1, \ldots, \tilde{\gamma}^r, \tilde{\delta}^1, \ldots, \tilde{\delta}^r, \tilde{\zeta}^1, \ldots, \tilde{\zeta}^s, \tilde{\eta}^1, \ldots, \tilde{\eta}^s)$ of (DI) such that for some $i \in \underline{r}$,

$$\varphi_{\mathrm{I}i}(\tilde{x}, \tilde{z}) - \varphi_{\mathrm{I}i}(x^*, z^*) > C\big[\varphi_{\mathrm{I}j}(x^*, z^*) - \varphi_{\mathrm{I}j}(\tilde{x}, \tilde{z})\big]$$

for all $C > 0$ and all $j \in \underline{r}$ such that $\varphi_{\mathrm{I}j}(x^*, z^*) > \varphi_{\mathrm{I}j}(\tilde{x}, \tilde{z})$. Now using $\varphi_{\mathrm{I}i} = \Phi_i/\Gamma_i$ and rearranging the above inequality, we get

$$\begin{aligned}&\Gamma_i(\tilde{x}, \tilde{u})\Phi_i(x^*, u^*) - \Phi_i(\tilde{x}, \tilde{u})\Gamma_i(x^*, u^*)\\ &\quad < -C\Gamma_i(\tilde{x}, \tilde{u})\Gamma_i(x^*, u^*)\big[\Phi_j(x^*, u^*)/\Gamma_j(x^*, u^*) - \Phi_j(\tilde{x}, \tilde{u})/\Gamma_j(\tilde{x}, \tilde{u})\big].\end{aligned}$$

Because the right-hand side of this inequality is negative and $\tilde{\lambda} > 0$, it follows that

$$\sum_{i=1}^{r} \tilde{\lambda}_i\big[\Gamma_i(\tilde{x}, \tilde{u})\Phi_i(x^*, u^*) - \Phi_i(\tilde{x}, \tilde{u})\Gamma_i(x^*, u^*)\big] < 0,$$

which contradicts (4.12) (with $(\bar{x}, \bar{u})$ replaced by (x^*, u^*) and (x, z) by $(\tilde{x}, \tilde{z})$). Therefore, we conclude that (x^*, z^*) is a properly efficient solution of (DI). □

We also have the following converse duality result for (P)–(D).

Theorem 4.3 (Strict converse duality). *Let $(\bar{x}, \bar{u})$ and $(x, z) \equiv (x, u, \lambda, v, w, \alpha^1, \ldots, \alpha^r, \beta^1, \ldots, \beta^r, \gamma^1, \ldots, \gamma^r, \delta^1, \ldots, \delta^r, \zeta^1, \ldots, \zeta^s, \eta^1, \ldots, \eta^s)$ be feasible solutions of* (P) *and* (DI)*, respectively, such that*

$$\sum_{i=1}^{r} \lambda_i\big[\Gamma_i(x, u)\Phi_i(\bar{x}, \bar{u}) - \Phi_i(x, u)\Gamma_i(\bar{x}, \bar{u})\big] \leqq 0. \tag{4.13}$$

Further, assume that $f_i(.,.,t)$ or $-g_i(.,.,t)$ is strictly convex throughout $[a, b]$ for at least one index $i \in \underline{r}$, or $h_j(.,.,t)$ is strictly convex throughout $[a, b]$ for at least one $j \in \underline{s}$ with the corresponding component w_j of w positive on $[a, b]$. Then $(\bar{x}(t), \bar{u}(t)) = (x(t), u(t))$ for all $t \in [a, b]$.

Proof. Suppose to the contrary that $(\bar{x}(t), \bar{u}(t)) \neq (x(t), u(t))$ on a subset of $[a, b]$ with positive length. Then proceeding as in the proof of Theorem 4.1 and using our strict convexity hypotheses, we arrive at the strict inequality

$$\sum_{i=1}^{r} \lambda_i\big[\Gamma_i(x, u)\Phi_i(\bar{x}, \bar{u}) - \Phi_i(x, u)\Gamma_i(\bar{x}, \bar{u})\big] > 0,$$

which contradicts (4.13). Hence we conclude that $(\bar{x}(t), \bar{u}(t)) = (x(t), u(t))$ for all $t \in [a, b]$. □

We observe that (DI) has the same objective function as the primal problem (P). Dual problems of this kind have been investigated previously in the area of finite-dimensional single-objective fractional programming.

In the above dual problems, the constraints (4.7) and (4.8) are superfluous in the sense that their deletion will not invalidate the foregoing duality results. More precisely, it can be shown that the following reduced versions of (DI) and ($\widetilde{\mathrm{D}}$I) obtained by dropping (4.7) and (4.8), and modifying (4.4), $\Phi_i(x,u)$ and $\Gamma_i(x,u)$, $i \in \underline{r}$, are also dual problems for (P):

(EI) Maximize $\big(\Pi_1(x,u,\alpha^1,\beta^1)/\Psi_1(x,u,\gamma^1,\delta^1),\ldots,$
$\Pi_r(x,u,\alpha^r,\beta^r)/\Psi_r(x,u,\gamma^r,\delta^r)\big)^T$

subject to (4.1), (4.5), (4.6), (4.9), and

$$\begin{aligned}&\sum_{i=1}^{r}\lambda_i\Big\{\Psi_i(x,u,\gamma^i,\delta^i)\big[\nabla_1 f_i(x,u,t)+K_i(t)^T\alpha^i(t)\big]\\&\quad-\Pi_i(x,u,\alpha^i,\beta^i)\big[\nabla_1 g_i(x,u,t)-M_i(t)^T\gamma^i(t)\big]\Big\}+A(t)^Tv(t)\\&\quad+\sum_{j=1}^{s}w_j(t)\big[\nabla_1 h_j(x,u,t)+P_j(t)^T\zeta^j(t)\big]+Dv(t)=0,\\&\quad t\in[a,b],\end{aligned}\tag{4.14}$$

$$\begin{aligned}&\sum_{i=1}^{r}\lambda_i\Big\{\Psi_i(x,u,\gamma^i,\delta^i)\big[\nabla_2 f_i(x,u,t)+L_i(t)^T\beta^i(t)\big]\\&\quad-\Phi_i(x,u,\alpha^i,\beta^i)\big[\nabla_2 g_i(x,u,t)-N_i(t)^T\delta^i(t)\big]\Big\}+B(t)^Tv(t)\\&\quad+\sum_{j=1}^{s}w_j(t)\big[\nabla_2 h_j(x,u,t)+Q_j(t)^T\eta^j(t)\big]=0,\quad t\in[a,b],\end{aligned}\tag{4.15}$$

$$\begin{aligned}&v(t)^T\big[-Dx+A(t)x+B(t)u\big]+\sum_{j=1}^{s}w_j(t)\big[h_j(x,u,t)+\zeta^j(t)^TP_j(t)x\\&\quad+\eta^j(t)^TQ_j(t)x\big]\geqq 0,\quad t\in[a,b],\end{aligned}\tag{4.16}$$

where

$$\Pi_i(x,u,\alpha^i,\beta^i)=\int_a^b\big[f_i(x,u,t)+\alpha^i(t)^TK_i(t)x+\beta^i(t)^TL_i(t)u\big]\,dt$$

and

$$\Psi_i(x,u,\gamma^i,\delta^i)=\int_a^b\big[g_i(x,u,t)-\gamma^i(t)^TM_i(t)x-\delta^i(t)^TN_i(t)u\big]\,dt,\quad i\in\underline{r};$$

($\widetilde{\mathrm{E}}$I) Maximize $\big(\Pi_1(x,u,\alpha^1,\beta^1)/\Psi_1(x,u,\gamma^1,\delta^1),\ldots,$
$\Pi_r(x,u,\alpha^r,\beta^r)/\Psi_r(x,u,\gamma^r,\delta^r)\big)^T$

subject to (4.1), (4.5), (4.6), (4.9), (4.16), and

$$\bigg\{\sum_{i=1}^{r}\lambda_i\Big\{\Psi_i(x,u,\gamma^i,\delta^i)\big[\nabla_1 f_i(x,u,t)^T+\alpha^i(t)^TK_i(t)\big]$$

$$- \Pi_i(x,u,\alpha^i,\beta^i)\big[\nabla_1 g_i(x,u,t)^T - \gamma^i(t)^T M_i(t)\big]\Big\} + v(t)^T A(t)$$

$$+ \sum_{j=1}^{s} w_j(t)\big[\nabla_1 h_j(x,u,t)^T + \zeta^j(t)^T P_j(t)\big] + Dv(t)^T\Bigg\}(\bar{x}-x) \geqq 0$$

for all $t \in [a,b]$ and all $\bar{x} \in C^n[a,b]$ such that $(\bar{x},u) \in \mathcal{F}$ for some $u \in \mathrm{PWS}^m[a,b]$,

$$\Bigg\{\sum_{i=1}^{r} \lambda_i \Big\{\Psi_i(x,u,\gamma^i,\delta^i)\big[\nabla_2 f_i(x,u,t)^T + \beta^i(t)^T L_i(t)\big]$$

$$- \Pi_i(x,u,\alpha^i,\beta^i)\big[\nabla_2 g_i(x,u,t)^T - \delta^i(t)^T N_i(t)\big]\Big\} + v(t)^T B(t)$$

$$+ \sum_{j=1}^{s} w_j(t)\big[\nabla_2 h_j(x,u,t)^T + \eta^j(t)^T Q_j(t)\big]\Bigg\}(\bar{u}-u) \geqq 0$$

for all $t \in [a,b]$ and all $\bar{u} \in \mathrm{PWS}^m[a,b]$ such that $(x,\bar{u}) \in \mathcal{F}$ for some $x \in C^n[a,b]$.

Since it may not be immediately apparent that (EI) and ($\widetilde{\mathrm{EI}}$) are dual problems for (P), we shall provide a proof for (P)–(EI).

Throughout this section and the next, it will be assumed that $\Pi_i(x,u,\alpha^i,\beta^i) \geqq 0$ *and* $\Psi_i(x,u,\gamma^i,\delta^i) > 0$ *for all* $x,u,\alpha^i,\beta^i,\gamma^i$, *and* δ^i, $i \in \underline{r}$, *such that* $(x,z) = (x,u,\lambda, v,w,\alpha^1,\dots,\alpha^r,\beta^1,\dots,\beta^r,\dots,\gamma^1,\dots,\gamma^r,\delta^1,\dots,\delta^r,\zeta^1,\dots,\zeta^s,\eta^1,\dots,\eta^s)$ *is a feasible solution of the dual problem under consideration.*

Theorem 4.4 (Weak duality). *Let* $(\bar{x},\bar{u})$ *and* (x,z), *defined above, be arbitrary feasible solutions of* (P) *and* (EI), *respectively. Then* $\theta(\bar{x},\bar{u}) \nleq \psi_{\mathrm{I}}(x,z)$, *where* $\psi_I = (\psi_{\mathrm{I}1},\dots,\psi_{\mathrm{I}r})^T$ *is the objective function of* (EI).

Proof. Since

$$\sum_{i=1}^{r} \lambda_i \big[\Psi_i(x,u,\gamma^i,\delta^i)\Phi_i(\bar{x},\bar{u}) - \Pi_i(x,u,\alpha^i,\beta^i)\Gamma_i(\bar{x},\bar{u})\big]$$

$$= \sum_{i=1}^{r} \lambda_i \Bigg\{\int_a^b \big[g_i(x,u,t) - \gamma^i(t)^T M_i(t)x - \delta^i(t)^T N_i(t)u\big]\,dt$$

$$\times \int_a^b \big[f_i(\bar{x},\bar{u},t) + \big\|K_i(t)\bar{x}\big\|_{k(i)} + \big\|L_i(t)\bar{u}\big\|_{\ell(i)}\big]\,dt$$

$$- \int_a^b \big[f_i(x,u,t) + \alpha^i(t)^T K_i(t)x + \beta^i(t)^T L_i(t)u\big]\,dt$$

$$\times \int_a^b \big[g_i(\bar{x},\bar{u},t) - \|M_i(t)\bar{x}\|_{m(i)} - \|N_i(t)\bar{u}\|_{n(i)}\big]\,dt\Bigg\}$$

$$= \sum_{i=1}^r \lambda_i \int_a^b \Big\{\Psi_i(x,u,\gamma^i,\delta^i)\big[f_i(\bar{x},\bar{u},t) - f_i(x,u,t)\big]$$
$$- \Pi_i(x,u,\alpha^i,\beta^i)\big[g_i(\bar{x},\bar{u},t) - g_i(x,u,t)\big] + \Psi_i(x,u,\gamma^i,\delta^i)\big[\|K_i(t)\bar{x}\|_{k(i)}$$
$$+ \|L_i(t)\bar{u}\|_{\ell(i)} - \alpha^i(t)^T K_i(t)x - \beta^i(t)^T L_i(t)u\big]$$
$$+ \Pi_i(x,u,\alpha^i,\beta^i)\big[\|M_i(t)\bar{x}\|_{m(i)} + \|N_i(t)\bar{u}\|_{n(i)}$$
$$- \gamma^i(t)^T M_i(t)x - \delta^i(t)^T N_i(t)u\big]\Big\}\,dt$$

$$\geqq \sum_{i=1}^r \lambda_i \int_a^b \Big\{\Psi_i(x,u,\gamma^i,\delta^i)\big[\nabla_1 f_i(x,u,t)^T(\bar{x}-x) + \nabla_2 f_i(x,u,t)^T(\bar{u}-u)\big]$$
$$- \Pi_i(x,u,\alpha^i,\beta^i)\big[\nabla_1 g_i(x,u,t)^T(\bar{x}-x) + \nabla_2 g_i(x,u,t)^T(\bar{u}-u)\big]$$
$$+ \Psi_i(x,u,\gamma^i,\delta^i)\big[\|K_i(t)\bar{x}\|_{k(i)} + \|L_i(t)\bar{u}\|_{\ell(i)} - \alpha^i(t)^T K_i(t)x$$
$$- \beta^i(t)^T L_i(t)u\big] + \Pi_i(x,u,\alpha^i,\beta^i)\big[\|M_i(t)\bar{x}\|_{m(i)} + \|N_i(t)\bar{u}\|_{n(i)}$$
$$- \gamma^i(t)^T M_i(t)x - \delta^i(t)^T N_i(t)u\big]\Big\}\,dt$$

(by the convexity of $f_i(.,.,t)$ and $-g_i(.,.,t)$, and nonnegativity of λ_i, $\Pi_i(x,u,\alpha^i,\beta^i)$, and $\Psi_i(x,u,\gamma^i,\delta^i)$, $i \in \underline{r}$)

$$= \int_a^b \Bigg\{\sum_{i=1}^r \lambda_i \Big\{\Psi_i(x,u,\gamma^i,\delta^i)\big[\|K_i(t)\bar{x}\|_{k(i)} + \|L_i(t)\bar{u}\|_{\ell(i)}$$
$$- \alpha^i(t)^T K_i(t)\bar{x} - \beta^i(t)^T L_i(t)\bar{u}\big] + \Pi_i(x,u,\alpha^i,\beta^i)\big[\|M_i(t)\bar{x}\|_{m(i)}$$
$$+ \|N_i(t)\bar{u}\|_{n(i)} - \gamma^i(t)^T M_i(t)\bar{x} - \delta^i(t)^T N_i(t)\bar{u}\big]\Big\}$$
$$- \Bigg\{v(t)^T A(t) + \sum_{j=1}^s w_j(t)\big[\nabla_1 h_j(x,u,t)^T + \zeta^j(t)^T P_j(t)\big] + Dv(t)^T\Bigg\}(\bar{x}-x)$$
$$- \Bigg\{v(t)^T B(t) + \sum_{j=1}^s w_j(t)\big[\nabla_2 h_j(x,u,t)^T + \eta^j(t)^T Q_j(t)\big]\Bigg\}(\bar{u}-u)\Bigg\}\,dt$$

(by (4.14) and (4.15))

$$\geqq \int_a^b \Bigg\{\sum_{i=1}^r \lambda_i \Big\{\Psi_i(x,u,\gamma^i,\delta^i)\big[\|K_i(t)\bar{x}\|_{k(i)} + \|L_i(t)\bar{u}\|_{\ell(i)}$$
$$- \|K_i(t)\bar{x}\|_{k(i)}\|\alpha^i(t)\|^*_{k(i)} - \|L_i(t)\bar{u}\|_{\ell(i)}\|\beta^i(t)\|^*_{\ell(i)}\big]$$

$$
\begin{aligned}
&+ \Pi_i(x,u,\alpha^i,\beta^i)\big[\|M_i(t)\bar{x}\|_{m(i)} + \|N_i(t)\bar{u}\|_{n(i)} - \|M_i(t)\bar{x}\|_{m(i)}\|\gamma^i(t)\|^*_{m(i)} \\
&- \|N_i(t)\bar{u}\|_{n(i)}\|\delta^i(t)\|^*_{n(i)}\big]\Big\} + \sum_{j=1}^{s} w_j(t)\big[h_j(x,u,t) - h_j(\bar{x},\bar{u},t) \\
&- \|P_j(t)\bar{x}\|_{p(j)}\|\zeta^j(t)\|^*_{p(j)} + \zeta^j(t)^T P_j(t)x - \|Q_j(t)\bar{u}\|_{q(j)}\|\eta^j(t)\|^*_{q(j)} \\
&+ \eta^j(t)^T Q_j(t)u\big] + v(t)^T\big[D(\bar{x}-x) - A(t)(\bar{x}-x) - B(t)(\bar{u}-u)\big]\Bigg\}\,dt \\
&\text{(by the convexity of } h_j(.,.,t),\ \text{nonnegativity of } \lambda_i,\ \Pi_i(x,u,\alpha^i,\beta^i), \\
&\Psi_i(x,u,\gamma^i,\delta^i),\ \text{and } w_j(t),\ i\in\underline{r},\ j\in\underline{s},\ (2.8),\ \text{and integration by parts)} \\
\geqq& \int_a^b \Bigg\{ v(t)^T\big[D\bar{x} - A(t)\bar{x} - B(t)\bar{u}\big] + v(t)^T\big[-Dx + A(t)x + B(t)u\big] \\
&+ \sum_{j=1}^{s} w_j(t)\big[h_j(x,u,t) + \zeta^j(t)^T P_j(t)x + \eta^j(t)^T Q_j(t)u\big] \\
&- \sum_{j=1}^{s} w_j(t)\big[h_j(\bar{x},\bar{u},t) + \|P_j(t)\bar{x}\|_{p(j)} + \|Q_j(t)\bar{u}\|_{q(j)}\big]\Bigg\}\,dt \\
&\text{(by the nonnegativity of } \lambda_i,\ \Pi_i(x,u,\alpha^i,\beta^i),\ \Psi_i(x,u,\gamma^i,\delta^i),\ i\in\underline{r}, \\
&\text{and } w(t),\ (4.5),\ \text{and } (4.6)) \\
\geqq& \ 0 \quad \text{(by the primal feasibility of } (\bar{x},\bar{u}),\ \text{nonnegativity of } w(t),\ \text{and } (4.16)),
\end{aligned}
$$

we conclude that

$$
\begin{aligned}
&\big(\Psi_1(x,u,\gamma^1,\delta^1)\Phi_1(\bar{x},\bar{u}) - \Pi_1(x,u,\alpha^1,\beta^1)\Gamma_1(\bar{x},\bar{u}),\ldots, \\
&\ \Psi_r(x,u,\gamma^r,\delta^r)\Phi_r(\bar{x},\bar{u}) - \Pi_r(x,u,\alpha^r,\beta^r)\Gamma_r(\bar{x},\bar{u})\big)^T \nleq (0,\ldots,0)^T,
\end{aligned}
$$

which implies that $\theta(\bar{x},\bar{u}) \nleq \psi_{\mathrm{I}}(x,z)$. □

Theorem 4.5 (Strong duality). *Let* (x^*,u^*) *be a properly efficient solution of* (P) *and assume that the constraints of* (P) *satisfy SCQ. Then there exist* $\lambda^*\in\Lambda$, $v^*\in \mathrm{PWS}^n[a,b]$, $w^*\in \mathrm{PWS}^s_+[a,b]$, $\alpha^{*i}\in \mathrm{PWS}^{k_i}[a,b]$, $\beta^{*i}\in \mathrm{PWS}^{\ell_i}[a,b]$, $\gamma^{*i}\in \mathrm{PWS}^{m_i}[a,b]$, $\delta^{*i}\in \mathrm{PWS}^{n_i}[a,b]$, $i\in\underline{r}$, $\zeta^{*j}\in \mathrm{PWS}^{p_j}[a,b]$ *and* $\eta^{*j}\in \mathrm{PWS}^{q_j}[a,b]$, $j\in\underline{s}$, *such that* $(x^*,z^*) \equiv (x^*,u^*,\lambda^*,v^*,w^*,\alpha^{*1},\ldots,\alpha^{*r},\beta^{*1},\ldots,\beta^{*r},\gamma^{*1},\ldots,\gamma^{*r},\delta^{*1},\ldots,\delta^{*r},\zeta^{*1},\ldots,\zeta^{*s},\eta^{*1},\ldots,\eta^{*s})$ *is a properly efficient solution of* (EI) *and* $\theta(x^*,u^*) = \psi_{\mathrm{I}}(x^*,z^*)$.

Proof. By Theorem 3.1, there exist $\lambda^*, v^*, w^*, \alpha^{*i}, \beta^{*i}, \gamma^{*i}, \delta^{*i}$, $i\in\underline{r}$, ζ^{*j} and η^{*j}, $j\in\underline{s}$, as specified above, such that (x^*,z^*) is a feasible solution of (EI) and $\theta(x^*,u^*) = \psi_{\mathrm{I}}(x^*,z^*)$ because the following relations hold for all $t\in[a,b]$:

$$
\begin{aligned}
&\alpha^{*i}(t)^T K_i(t)x^* = \|K_i(t)x^*\|_{k(i)}, \qquad \beta^{*i}(t)^T L_i(t)u^* = \|L_i(t)u^*\|_{\ell(i)}, \\
&\gamma^{*i}(t)^T M_i(t)x^* = \|M_i(t)x^*\|_{m(i)}, \qquad \delta^{*i}(t)^T M_i(t)u^* = \|N_i(t)u^*\|_{n(i)}, \quad i\in\underline{r},
\end{aligned}
$$

$$\zeta^{*j}(t)^T P_j(t)x^* = \left\| P_j(t)x^* \right\|_{p(j)}, \quad \eta^{*j}(t)^T Q_j(t)u^* = \left\| Q_j(t)u^* \right\|_{q(j)}, \quad j \in \underline{s}.$$

That (x^*, z^*) is properly efficient can be verified as in the proof of Theorem 4.2. □

Theorem 4.6 (Strict converse duality). *Let* $(\bar{x}, \bar{u})$ *and* $(x, z) = (x, u, \lambda, v, w, \alpha^1, \ldots, \alpha^r, \beta^1, \ldots, \beta^r, \gamma^1, \ldots, \gamma^r, \delta^1, \ldots, \delta^r, \zeta^1, \ldots, \zeta^s, \eta^1, \ldots, \eta^s)$ *be feasible solutions of* (P) *and* (EI)*, respectively, such that*

$$\sum_{i=1}^{r} \lambda_i \big[\Psi_i(x, u, \gamma^i, \delta^i)\Phi_i(\bar{x}, \bar{u}) - \Pi_i(x, u, \alpha^i, \beta^i)\Gamma_i(\bar{x}, \bar{u})\big] \leqq 0.$$

Further, assume that $f_i(.,.,t)$ *or* $-g_i(.,.,t)$ *is strictly convex throughout* $[a, b]$ *for at least one index* $i \in \underline{r}$*, or* $h_j(.,.,t)$ *is strictly convex throughout* $[a, b]$ *for at least one* $j \in \underline{s}$ *with the corresponding component* w_j *of* w *positive on* $[a, b]$*. Then* $(\bar{x}(t), \bar{u}(t)) = (x(t), u(t))$ *for all* $t \in [a, b]$.

Proof. The proof is similar to that of Theorem 4.3. □

5. Duality model II

Following the same pattern of presentation as in the preceding section, here we shall formulate and discuss four additional semiparametric duality models for (P). We begin with the following variants of (DI) and $(\widetilde{\mathrm{D}}\mathrm{I})$:

(DII) Maximize $\big(\big[\Phi_1(x, u) + \Omega(x, u, v, w)\big]/\Gamma_1(x, u), \ldots, \big[\Phi_r(x, u) + \Omega(x, u, v, w)\big]/\Gamma_r(x, u)\big)^T$

subject to (4.1), (4.4)–(4.9), and

$$\sum_{i=1}^{r} \lambda_i \Bigg\{ \Gamma_i(x, u) \Bigg\{ \nabla_1 f_i(x, u, t) + K_i(t)^T \alpha^i(t) + A(t)^T v(t) + \sum_{j=1}^{s} w_j(t)\big[\nabla_1 h_j(x, u, t) + P_j(t)^T \zeta^j(t)\big] + Dv(t) \Bigg\} - \big[\Phi_i(x, u) + \Omega(x, u, v, w)\big]\big[\nabla_1 g_i(x, u, t) - M_i(t)^T \gamma^i(t)\big] \Bigg\} = 0,$$
$$t \in [a, b], \tag{5.1}$$

$$\sum_{i=1}^{r} \lambda_i \Bigg\{ \Gamma_i(x, u) \Bigg\{ \nabla_2 f_i(x, u, t) + L_i(t)^T \beta^i(t) + B(t)^T v(t) + \sum_{j=1}^{s} w_j(t)\big[\nabla_2 h_j(x, u, t) + Q_j(t)^T \zeta^j(t)\big] \Bigg\}$$

$$- \big[\Phi_i(x,u) + \Omega(x,u,v,w)\big]\big[\nabla_2 g_i(x,u,t) - N_i(t)^T \delta^i(t)\big]\Bigg\} = 0,$$
$$t \in [a,b], \tag{5.2}$$

where

$$\Omega(x,u,v,w) = \int_a^b \Bigg\{ v(t)^T\big[-Dx + A(t)x + B(t)u\big] + \sum_{j=1}^{s} w_j(t)\big[h_j(x,u,t) + \|P_j(t)x\|_{p(j)} + \|Q_j(t)u\|_{q(j)}\big]\Bigg\}\,dt;$$

(D̃II) Maximize $\big(\big[\Phi_1(x,u) + \Omega(x,u,v,w)\big]/\Gamma_1(x,u), \ldots, \big[\Phi_r(x,u) + \Omega(x,u,v,w)\big]/\Gamma_r(x,u)\big)^T$

subject to (4.1), (4.4)–(4.9), and

$$\Bigg\{\sum_{i=1}^{r}\lambda_i\Bigg\{\Gamma_i(x,u)\Bigg\{\nabla_1 f_i(x,u,t)^T + \alpha^i(t)^T K_i(t) + v(t)^T A(t) + \sum_{j=1}^{s} w_j(t)\big[\nabla_1 h_j(x,u,t)^T + \zeta^j(t)^T P_j(t)\big] + Dv(t)^T\Bigg\}$$
$$- \big[\Phi_i(x,u) + \Omega(x,u,v,w)\big]\big[\nabla_1 g_i(x,u,t)^T - \gamma^i(t)^T M_i(t)\big]\Bigg\}\Bigg\}$$
$$\times (\bar{x} - x) \geqq 0 \quad \text{for all } t \in [a,b] \text{ and all } \bar{x} \in C^n[a,b] \text{ such that } (\bar{x},u) \in \mathcal{F} \text{ for some } u \in \mathrm{PWS}^m[a,b],$$

$$\Bigg\{\sum_{i=1}^{r}\lambda_i\Bigg\{\Gamma_i(x,u)\Bigg\{\nabla_2 f_i(x,u,t)^T + \beta^i(t)^T L_i(t) + v(t)^T B(t) + \sum_{j=1}^{s} w_j(t)\big[\nabla_2 h_j(x,u,t)^T + \eta^j(t)^T Q_j(t)\big]\Bigg\}$$
$$- \big[\Phi_i(x,u) + \Omega(x,u,v,w)\big]\big[\nabla_2 g_i(x,u,t)^T - \delta^i(t)^T N_i(t)\big]\Bigg\}\Bigg\}$$
$$\times (\bar{u} - u) \geqq 0 \quad \text{for all } t \in [a,b] \text{ and all } \bar{u} \in \mathrm{PWS}^m[a,b] \text{ such that } (x,\bar{u}) \in \mathcal{F} \text{ for some } x \in C^n[a,b].$$

The remarks made earlier concerning the relationships between (DI) and (D̃I) are, of course, also applicable to (DII) and (D̃II).

We now proceed to state and prove weak and strong duality theorems for (P)–(DII).

Theorem 5.1 (Weak duality). *Let* $(\bar{x}, \bar{u})$ *and* $(x, z) \equiv (x, u, \lambda, v, w, \alpha^1, \ldots, \alpha^r, \beta^1, \ldots, \beta^r, \gamma^1, \ldots, \gamma^r, \delta^1, \ldots, \delta^r, \zeta^1, \ldots, \zeta^s, \eta^1, \ldots, \eta^s)$ *be arbitrary feasible solutions of* (P) *and* (DII), *respectively. Then* $\theta(\bar{x}, \bar{u}) \nleq \varphi_{\mathrm{II}}(x, z)$, *where* $\varphi_{\mathrm{II}} = (\varphi_{\mathrm{II}1}, \ldots, \varphi_{\mathrm{II}r})^T$ *is the objective function of* (DII).

Proof. Since

$$\begin{aligned}
&\sum_{i=1}^{r} \lambda_i \bigl\{ \Gamma_i(x, u) \Phi_i(\bar{x}, \bar{u}) - \bigl[\Phi_i(x, u) + \Omega(x, u, v, w)\bigr] \Gamma_i(\bar{x}, \bar{u}) \bigr\} \\
&\quad = \sum_{i=1}^{r} \lambda_i \Bigl\{ \Gamma_i(x, u)\bigl[\Phi_i(\bar{x}, \bar{u}) - \Phi_i(x, u)\bigr] - \Phi_i(x, u)\bigl[\Gamma_i(\bar{x}, \bar{u}) - \Gamma_i(x, u)\bigr] \\
&\qquad - \Omega(x, u, v, w) \Gamma_i(\bar{x}, \bar{u}) \Bigr\} \\
&\quad = \sum_{i=1}^{r} \lambda_i \left\{ \int_a^b \Bigl\{ \Gamma_i(x, u)\bigl[f_i(\bar{x}, \bar{u}, t) - f_i(x, u, t) + \bigl\|K_i(t)\bar{x}\bigr\|_{k(i)} + \bigl\|L_i(t)\bar{u}\bigr\|_{\ell(i)} \right. \\
&\qquad - \bigl\|K_i(t)x\bigr\|_{k(i)} - \bigl\|L_i(t)u\bigr\|_{\ell(i)}\bigr] - \Phi_i(x, u)\bigl[g_i(\bar{x}, \bar{u}, t) - g_i(x, u, t) \\
&\qquad - \bigl\|M_i(t)\bar{x}\bigr\|_{m(i)} - \bigl\|N_i(t)\bar{u}\bigr\|_{n(i)} + \bigl\|M_i(t)x\bigr\|_{m(i)} + \bigl\|N_i(t)u\bigr\|_{n(i)}\bigr] \Bigr\}\, dt \\
&\qquad \left. - \Omega(x, u, v, w) \Gamma_i(\bar{x}, \bar{u}) \right\} \\
&\quad \geqq \sum_{i=1}^{r} \lambda_i \left\{ \int_a^b \Bigl\{ \Gamma_i(x, u)\bigl[\nabla_1 f_i(x, u, t)^T(\bar{x} - x) + \nabla_2 f_i(x, u, t)^T(\bar{u} - u) \right. \\
&\qquad + \bigl\|K_i(t)\bar{x}\bigr\|_{k(i)} + \bigl\|L_i(t)\bar{u}\bigr\|_{\ell(i)} - \bigl\|K_i(t)x\bigr\|_{k(i)} - \bigl\|L_i(t)u\bigr\|_{\ell(i)}\bigr] \\
&\qquad - \Phi_i(x, u)\bigl[\nabla_1 g_i(x, u, t)^T(\bar{x} - x) + \nabla_2 g_i(x, u, t)^T(\bar{u} - u) \\
&\qquad - \bigl\|M_i(t)\bar{x}\bigr\|_{m(i)} - \bigl\|N_i(t)\bar{u}\bigr\|_{n(i)} + \bigl\|M_i(t)x\bigr\|_{m(i)} + \bigl\|N_i(t)u\bigr\|_{n(i)}\bigr] \Bigr\}\, dt \\
&\qquad \left. - \Omega(x, u, v, w) \Gamma_i(\bar{x}, \bar{u}) \right\} \quad \text{(by the convexity of } f_i(.,.,t) \text{ and } -g_i(.,.,t), \\
&\qquad \text{and nonnegativity of } \lambda_i,\ \Phi_i(x, u),\ \text{and } \Gamma_i(x, u),\ i \in \underline{r}) \\
&\quad = \sum_{i=1}^{r} \lambda_i \left\{ \int_a^b \left\{ \Gamma_i(x, u) \left\{ \bigl\|K_i(t)\bar{x}\bigr\|_{k(i)} + \bigl\|L_i(t)\bar{u}\bigr\|_{\ell(i)} - \bigl\|K_i(t)x\bigr\|_{k(i)} \right.\right.\right. \\
&\qquad - \bigl\|L_i(t)u\bigr\|_{\ell(i)} - \alpha^i(t)^T K_i(t)(\bar{x} - x) - \beta^i(t)^T L_i(t)(\bar{u} - u) - Dv(t)^T(\bar{x} - x) \\
&\qquad - v(t)^T\bigl[A(t)(\bar{x} - x) + B(t)(\bar{u} - u)\bigr] - \sum_{j=1}^{s} w_j(t)\bigl[\nabla_1 h_j(x, u, t)^T(\bar{x} - x)
\end{aligned}$$

$$+ \nabla_2 h_j(x,u,t)^T(\bar{u}-u) + \zeta^j(t)^T P_j(t)(\bar{x}-x) + \eta^j(t)^T Q_j(t)(\bar{u}-u)\big]\Big\}$$

$$- \Phi_i(x,u)\big[\|M_i(t)x\|_{m(i)} + \|N_i(t)u\|_{n(i)} - \|M_i(t)\bar{x}\|_{m(i)} - \|N_i(t)\bar{u}\|_{n(i)}$$

$$+ \gamma^i(t)^T M_i(t)(\bar{x}-x) + \delta^i(t)^T N_i(t)(\bar{u}-u)\big]$$

$$+ \Omega(x,u,v,w)\big[\nabla_1 g_i(x,u,t)^T(\bar{x}-x) + \nabla_2 g_i(x,u,t)^T(\bar{u}-u)$$

$$- \delta^i(t)^T M_i(t)(\bar{x}-x) - \delta^i(t)^T N_i(t)(\bar{u}-u)\big]\Big\}\, dt - \Omega(x,u,v,w)\Gamma_i(\bar{x},\bar{u})\Big\}$$

(by (5.1) and (5.2))

$$\geqq \sum_{i=1}^r \lambda_i \Big\{\int_a^b \Big\{\Gamma_i(x,u)\Big\{\big[\|K_i(t)\bar{x}\|_{k(i)} + \|L_i(t)\bar{u}\|_{\ell(i)} - \|K_i(t)\bar{x}\|_{k(i)}\|\alpha^i(t)\|^*_{k(i)}$$

$$- \|L_i(t)\bar{u}\|_{\ell(i)}\|\beta^i(t)\|^*_{\ell(i)}\big] + v(t)^T\big[D\bar{x} - Dx - A(t)(\bar{x}-x) - B(t)(\bar{u}-u)\big]$$

$$+ \sum_{j=1}^s w_j(t)\big[h_j(x,u,t) - h_j(\bar{x},\bar{u},t) + \zeta^j(t)^T P_j(t)x$$

$$- \|P_j(t)\bar{x}\|_{p(j)}\|\zeta^j(t)\|^*_{p(j)} + \eta^j(t)^T Q_j(t)u - \|Q_j(t)\bar{u}\|_{q(j)}\|\eta^j(t)\|^*_{q(j)}\big]\Big\}$$

$$- \Phi_i(x,u)\big[-\|M_i(t)\bar{x}\|_{m(i)} - \|N_i(t)\bar{u}\|_{n(i)} + \|M_i(t)\bar{x}\|_{m(i)}\|\gamma^i(t)\|^*_{m(i)}$$

$$+ \|N_i(t)\bar{u}\|_{n(i)}\|\delta^i(t)\|^*_{n(i)}\big] + \Omega(x,u,v,w)\big[g_i(\bar{x},\bar{u},t) - g_i(x,u,t)$$

$$+ \gamma^i(t)^T M_i(t)x - \|M_i(t)\bar{x}\|_{m(i)}\|\gamma^i(t)\|^*_{m(i)} + \delta^i(t)^T N_i(t)u$$

$$- \|N_i(t)\bar{u}\|_{n(i)}\|\delta^i(t)\|^*_{n(i)}\big]\Big\}\, dt - \Omega(x,u,v,w)\Gamma_i(\bar{x},\bar{u})\Big\}$$

(by the convexity of $-g_i(.,.,t)$ and $h_j(.,.,t)$, nonnegativity of λ_i, $\Phi_i(x,u)$, $\Gamma_i(x,u)$, $w_j(t)$, $i \in \underline{r}$, $j \in \underline{s}$, and $\Omega(x,u,v,w)$ ((4.4)), (2.8), (4.7), and integradon by parts)

$$\geqq \sum_{i=1}^r \lambda_i \Big\{\int_a^b \Big\{\Gamma_i(x,u)\Big\{v(t)^T\big[-Dx + A(t)x + B(t)u\big]$$

$$+ \sum_{j=1}^s w_j(t)\big[h_j(x,u,t) + \zeta^j(t)^T P_j(t)x + \eta^j(t)^T Q_j(t)u\big]\Big\}$$

$$+ \Omega(x,u,v,w)\Big\{g_i(\bar{x},\bar{u},t) - \|M_i(t)\bar{x}\|_{m(i)} - \|N_i(t)\bar{u}\|_{n(i)}$$

$$- \big[g_i(x,u,t) - \|M_i(t)\bar{x}\|_{m(i)} - \|N_i(t)\bar{u}\|_{n(i)}\big]\Big\}\Big\}\, dt - \Omega(x,u,v,w)\Gamma_i(\bar{x},\bar{u})\Big\}$$

$$
\begin{aligned}
&\text{(by the nonnegativity of } \lambda_i,\ \Phi_i(x,u),\ \Gamma_i(x,u),\ i \in \underline{r},\ w_j(t),\ j \in \underline{s}, \text{ and}\\
&\Omega(x,u,v,w)\ ((4.4)),\ \text{primal feasibility of } (\bar{x},\bar{u}),\ \text{and (4.5)–(4.7))}\\
&= \sum_{i=1}^{r} \lambda_i \big[\Gamma_i(x,u) + \Gamma_i(\bar{x},\bar{u}) - \Gamma_i(x,u) - \Gamma_i(\bar{x},\bar{u})\big]\Omega(x,u,v,w) \quad \text{(by (4.8))}\\
&= 0,
\end{aligned}
$$

we conclude that

$$
\begin{aligned}
&\big(\Gamma_1(x,u)\Phi_1(\bar{x},\bar{u}) - \big[\Phi_1(x,u) + \Omega(x,u,v,w)\big]\Gamma_1(\bar{x},\bar{u}), \ldots,\\
&\quad \Gamma_r(x,u)\Phi_r(\bar{x},\bar{u}) - \big[\Phi_r(x,u) + \Omega(x,u,v,w)\big]\Gamma_r(\bar{x},\bar{u})\big)^T \nleq (0,\ldots,0)^T,
\end{aligned}
$$

which implies that $\theta(\bar{x},\bar{u}) \nleq \varphi_{\mathrm{II}}(x,z)$. $\square$

Theorem 5.2 (Strong duality). *Let (x^*,u^*) be a properly efficient solution of* (P) *and assume that the constraints of* (P) *satisfy SCQ. Then there exist $\lambda^* \in \Lambda$, $v^* \in \mathrm{PWS}^n[a,b]$, $w^* \in \mathrm{PWS}^s_+[a,b]$, $\alpha^{*i} \in \mathrm{PWS}^{k_i}[a,b]$, $\beta^{*i} \in \mathrm{PWS}^{\ell_i}[a,b]$, $\gamma^{*i} \in \mathrm{PWS}^{m_i}[a,b]$, $\delta^{*i} \in \mathrm{PWS}^{n_i}[a,b]$, $i \in \underline{r}$, $\zeta^{*j} \in \mathrm{PWS}^{p_j}[a,b]$ and $\eta^{*j} \in \mathrm{PWS}^{q_j}[a,b]$, $j \in \underline{s}$, such that $(x^*,z^*) \equiv (x^*,u^*,\lambda^*,v^*,w^*,\alpha^{*1},\ldots,\alpha^{*r},\beta^{*1},\ldots,\beta^{*r},\gamma^{*1},\ldots,\gamma^{*r},\delta^{*1},\ldots,\delta^{*r},\zeta^{*1},\ldots,\zeta^{*s},$ $\eta^{*1},\ldots,\eta^{*s})$ is a properly efficient solution of* (DII) *and $\theta(x^*,u^*) = \varphi_{\mathrm{II}}(x^*,z^*)$.*

Proof. By Theorem 3.1, there exist $\lambda^*, \alpha^{*i}, \beta^{*i}, \gamma^{*i}, \delta^{*i}$, $i \in \underline{r}$, ζ^{*j} and η^{*j}, $j \in \underline{s}$, as specified above, $v^0 \in \mathrm{PWS}^n[a,b]$, and $w^0 \in \mathrm{PWS}^s_+[a,b]$ such that the following relations hold for all $t \in [a,b]$:

$$
\begin{aligned}
&\sum_{i=1}^{r} \lambda_i^* \Big\{\Gamma_i(x^*,u^*)\big[\nabla_1 f_i(x^*,u^*,t) + K_i(t)^T \alpha^{*i}(t)\big]\\
&\quad - \Phi_i(x^*,u^*)\big[\nabla_1 g_i(x^*,u^*,t) - M_i(t)^T \gamma^{*i}(t)\big]\Big\} + A(t)^T v^0(t)\\
&\quad + \sum_{j=1}^{s} w_j^0(t)\big[\nabla_1 h_j(x^*,u^*,t) + P_j(t)^T \zeta^{*j}(t)\big] + D v^0(t) = 0,
\end{aligned} \tag{5.3}
$$

$$
\begin{aligned}
&\sum_{i=1}^{r} \lambda_i^* \Big\{\Gamma_i(x^*,u^*)\big[\nabla_2 f_i(x^*,u^*,t) + L_i(t)^T \beta^{*i}(t)\big]\\
&\quad - \Phi_i(x^*,u^*)\big[\nabla_2 g_i(x^*,u^*,t) - N_i(t)^T \gamma^{*i}(t)\big]\Big\} + B(t)^T v^0(t)\\
&\quad + \sum_{j=1}^{s} w_j^0(t)\big[\nabla_2 h_j(x^*,u^*,t) + Q_j(t)^T \eta^{*j}(t)\big] = 0,
\end{aligned} \tag{5.4}
$$

$$
\sum_{i=1}^{r} w_j^0(t)\big[h_j(x^*,u^*,t) + \big\|P_j(t)x^*\big\|_{p(j)} + \big\|Q_j(t)u^*\big\|_{q(j)}\big] = 0, \tag{5.5}
$$

$$
\begin{aligned}
&\big\|\alpha^{*i}(t)\big\|^*_{k(i)} \leqq 1, \quad \big\|\beta^{*i}(t)\big\|^*_{\ell(i)} \leqq 1, \quad \big\|\gamma^{*i}(t)\big\|^*_{m(i)} \leqq 1,\\
&\big\|\delta^{*i}(t)\big\|^*_{n(i)} \leqq 1, \quad i \in \underline{r},
\end{aligned} \tag{5.6}
$$

$$\left\|\zeta^{*j}(t)\right\|_{p(j)}^{*} \leqq 1, \quad \left\|\eta^{*j}(t)\right\|_{q(j)}^{*} \leqq 1, \quad j \in \underline{s}, \tag{5.7}$$

$$\begin{aligned} &\alpha^{*i}(t)^T K_i(t)x^* = \left\|K_i(t)x^*\right\|_{k(i)}, \quad \beta^{*i}(t)^T L_i(t)u^* = \left\|L_i(t)u^*\right\|_{\ell(i)}, \\ &\gamma^{*i}(t)^T M_i(t)x^* = \left\|M_i(t)x^*\right\|_{m(i)}, \quad \delta^{*i}(t)^T N_i(t)u^* = \left\|N_i(t)u^*\right\|_{n(i)}, \\ &\quad i \in \underline{r}, \end{aligned} \tag{5.8}$$

$$\begin{aligned} &\zeta^{*j}(t)^T P_j(t)x^* = \left\|P_j(t)x^*\right\|_{p(j)}, \quad \eta^{*j}(t)^T Q_j(t)u^* = \left\|Q_j(t)u^*\right\|_{q(j)}, \\ &\quad j \in \underline{s}. \end{aligned} \tag{5.9}$$

Since $c \in \sum_{i=1}^{r} \lambda_i^* \Gamma_i(x^*, u^*) > 0$, (5.3)–(5.5) can be expressed as

$$\begin{aligned} &\sum_{i=1}^{r} \lambda_i^* \Biggl\{ \Gamma_i(x^*, u^*) \Biggl\{ \nabla_1 f_i(x^*, u^*, t) + K_i(t)^T \alpha^{*i}(t) + A(t)^T \bigl[v^0(t)/c\bigr] \\ &\quad + \sum_{j=1}^{s} \bigl[w_j^0(t)/c\bigr]\bigl[\nabla_1 h_j(x^*, u^*, t) + P_j(t)^T \zeta^{*j}(t)\bigr] + D\bigl[v^0(t)/c\bigr] \Biggr\} \\ &\quad - \Phi_i(x^*, u^*)\bigl[\nabla_1 g_i(x^*, u^*, t) - M_i(t)^T \gamma^{*i}(t)\bigr] \Biggr\} = 0, \end{aligned} \tag{5.10}$$

$$\begin{aligned} &\sum_{i=1}^{r} \lambda_i^* \Biggl\{ \Gamma_i(x^*, u^*) \Biggl\{ \nabla_2 f_i(x^*, u^*, t) + L_i(t)^T \beta^{*i}(t) + B(t)^T \bigl[v^0(t)/c\bigr] \\ &\quad + \sum_{j=1}^{s} \bigl[w_j^0(t)/c\bigr]\bigl[\nabla_2 h_j(x^*, u^*, t) + Q_j(t)^T \eta^{*j}(t)\bigr] \Biggr\} \\ &\quad - \Phi_i(x^*, u^*)\bigl[\nabla_2 g_i(x^*, u^*, t) - N_i(t)^T \gamma^{*i}(t)\bigr] \Biggr\} = 0, \end{aligned} \tag{5.11}$$

$$\sum_{j=1}^{s} \bigl[w_j^0(t)/c\bigr]\bigl[h_j(x^*, u^*, t) + \left\|P_j(t)x^*\right\|_{p(j)} + \left\|Q_j(t)u^*\right\|_{q(j)}\bigr] = 0. \tag{5.12}$$

Now letting $v^* = v^0/c$ and $w^* = w^0/c$, and noting that $\Omega(x^*, u^*, v^*, w^*) = 0$, (5.10)–(5.12) can be rewritten as

$$\begin{aligned} &\sum_{i=1}^{r} \lambda_i^* \Biggl\{ \Gamma_i(x^*, u^*) \Biggl\{ \nabla_1 f_i(x^*, u^*, t) + K_i(t)^T \alpha^{*i}(t) + A(t)^T v^*(t) \\ &\quad + \sum_{j=1}^{s} w_j^*(t)\bigl[\nabla_1 h_j(x^*, u^*, t) + P_j(t)^T \zeta^{*j}(t)\bigr] + Dv^*(t) \Biggr\} \\ &\quad - \bigl[\Phi_i(x^*, u^*) + \Omega(x^*, u^*, v^*, w^*)\bigr]\bigl[\nabla_1 g_i(x^*, u^*, t) - M_i(t)^T \gamma^{*i}(t)\bigr] \Biggr\} = 0, \\ &\quad t \in [a, b], \end{aligned} \tag{5.13}$$

$$\sum_{i=1}^{r} \lambda_i^* \Bigg\{ \Gamma_i(x^*, u^*) \Bigg\{ \nabla_2 f_i(x^*, u^*, t) + L_i(t)^T \beta^{*i}(t) + B(t)^T v^*(t)$$
$$+ \sum_{j=1}^{s} w_j^*(t) \big[\nabla_2 h_j(x^*, u^*, t) + Q_j(t)^T \eta^{*j}(t) \big] \Bigg\}$$
$$- \big[\Phi_i(x^*, u^*) + \Omega(x^*, u^*, v^*, w^*) \big] \big[\nabla_2 g_i(x^*, u^*, t) - N_i(t)^T \delta^{*i}(t) \big] \Bigg\} = 0,$$
$$t \in [a, b], \tag{5.14}$$
$$\sum_{j=1}^{s} w_j^*(t) \big[h_j(x^*, u^*, t) + \big\| P_j(t) x^* \big\|_{p(j)} + \big\| Q_j(t) u^* \big\|_{q(j)} \big] = 0,$$
$$t \in [a, b]. \tag{5.15}$$

It is clear from (5.6)–(5.9) and (5.13)–(5.15) that (x^*, z^*) is a feasible solution of (DII) and $\theta(x^*, u^*) = \varphi_{\text{II}}(x^*, z^*)$. The fact that (x^*, z^*) is properly efficient can be established in exactly the same manner as in the proof of Theorem 4.2. □

Theorem 5.3 (Strict converse duality). *Let* $(\bar{x}, \bar{u})$ *and* $(x, u, \lambda, v, w, \alpha^1, \ldots, \alpha^r, \beta^1, \ldots, \beta^r, \gamma^1, \ldots, \gamma^r, \delta^1, \ldots, \delta^r, \zeta^1, \ldots, \zeta^s, \eta^1, \ldots, \eta^s)$ *be feasible solutions of* (P) *and* (DII), *respectively, such that*

$$\sum_{i=1}^{r} \lambda_i \big\{ \Gamma_i(x, u) \Phi_i(\bar{x}, \bar{u}) - \big[\Phi_i(x, u) + \Omega(x, u, v, w) \big] \Gamma_i(\bar{x}, \bar{u}) \big\} \leqq 0.$$

Further, assume that $f_i(.,.,t)$ *or* $-g_i(.,.,t)$ *is strictly convex throughout* $[a, b]$ *for at least one index* $i \in \underline{r}$, *or* $h_j(.,.,t)$ *is strictly convex throughout* $[a, b]$ *for at least one* $j \in \underline{s}$ *with the corresponding component* w_j *of* w *positive on* $[a, b]$. *Then* $(\bar{x}(t), \bar{u}(t)) = (x(t), u(t))$ *for all* $t \in [a, b]$.

Proof. The proof is similar to that of Theorem 4.3. □

We next turn to a brief discussion of the reduced versions of (DII) and ($\widetilde{\text{D}}$II) obtained by deleting the constraints (4.7) and (4.8), and altering the objective functions and remaining constraints accordingly. These streamlined variants of (DII) and ($\widetilde{\text{D}}$II) take the following forms:

(EII) Maximize $\big(\big[\Pi_1(x, u, \alpha^1, \beta^1) + \Delta(x, u, v, w, \zeta, \eta) \big] / \Psi_1(x, u, \gamma^1, \delta^1), \ldots,$
$\big[\Pi_r(x, u, \alpha^r, \beta^r) + \Delta(x, u, v, w, \zeta, \eta) \big] / \big[\Psi_r(x, u, \gamma^r, \delta^r) \big)^T$

subject to (4.1), (4.5), (4.6), (4.9), (4.16), and

$$\sum_{i=1}^{r} \lambda_i \Bigg\{ \Psi_i(x, u, \gamma^i, \delta^i) \Bigg\{ \nabla_1 f_i(x, u, t) + K_i(t)^T \alpha^i(t) + A(t)^T v(t)$$
$$+ \sum_{j=1}^{s} w_j(t) \big[\nabla_1 h_j(x, u, t) + P_j(t)^T \zeta^j(t) \big] + D v(t) \Bigg\}$$

$$- \big[\Pi_i(x,u,\alpha^i,\beta^i) + \Delta(x,u,v,w,\zeta,\eta)\big]\big[\nabla_1 g_i(x,u,t) - M_i(t)^T\gamma^i(t)\big]\Bigg\}$$

$$= 0, \quad t \in [a,b],$$

$$\sum_{i=1}^{r} \lambda_i \Bigg\{\Psi_i(x,u,\gamma^i,\delta^i)\Bigg\{\nabla_2 f_i(x,u,t) + L_i(t)^T\beta^i(t) + B(t)^T v(t)$$

$$+ \sum_{j=1}^{s} w_j(t)\big[\nabla_2 h_j(x,u,t) + Q_j(t)^T\eta^j(t)\big]\Bigg\}$$

$$- \big[\Pi_i(x,u,\alpha^i,\beta^i) + \Delta(x,u,v,w,\zeta,\eta)\big]\big[\nabla_2 g_i(x,u,t) - N_i(t)^T\zeta^i(t)\big]\Bigg\}$$

$$= 0, \quad t \in [a,b],$$

where

$$\Delta(x,u,v,w,\zeta,\eta) = \int_a^b \Bigg\{v(t)^T\big[-Dx + A(t)x + B(t)u\big]$$

$$+ \sum_{j=1}^{s} w_j(t)\big[h_j(x,u,t) + \zeta^j(t)^T P_j(t)x + \eta^j(t)^T Q_j(t)u\big]\Bigg\}\,dt,$$

and Π_i and Ψ_i, $i \in \underline{r}$, are as defined in Section 4;

($\widetilde{\mathrm{E}}$II) Maximize $\big(\big[\Pi_1(x,u,\alpha^1,\beta^1) + \Delta(x,u,v,w,\zeta,\eta)\big]\big/\big[\Psi_1(x,u,\gamma^1,\delta^1),\ldots,$
$\big[\Pi_r(x,u,\alpha^r,\beta^r) + \Delta(x,u,v,w,\zeta,\eta)\big]\big/\Psi_r(x,u,\gamma^r,\delta^r)\big)^T$

subject to (4.1), (4.5), (4.6), (4.9), (4.16), and

$$\Bigg\{\sum_{i=1}^{r} \lambda_i \Bigg\{\Psi_i(x,u,\gamma^i,\delta^i)\Bigg\{\nabla_1 f_i(x,u,t)^T + \alpha^i(t)^T K_i(t) + v(t)^T A(t)$$

$$+ \sum_{j=1}^{s} w_j(t)\big[\nabla_1 h_j(x,u,t)^T + \zeta^j(t)^T P_j(t)\big] + Dv(t)^T\Bigg\}$$

$$- \big[\Pi_i(x,u,\alpha^i,\beta^i) + \Delta(x,u,v,w,\zeta,\eta)\big]\big[\nabla_1 g_i(x,u,t)^T$$

$$- \gamma^i(t)^T M_i(t)\big]\Bigg\}\Bigg\}(\bar{x} - x) \geqq 0, \quad \text{for all } t \in [a,b] \text{ and all } \bar{x} \in C^n[a,b]$$

such that $(\bar{x},u) \in \mathcal{F}$ for some $u \in \mathrm{PWS}^m[a,b]$,

$$\Bigg\{\sum_{i=1}^{r} \lambda_i \Bigg\{\Psi_i(x,u,\gamma^i,\delta^i)\Bigg\{\nabla_2 f_i(x,u,t)^T + \beta^i(t)^T L_i(t) + v(t)^T B(t)$$

$$+ \sum_{j=1}^{s} w_j(t)\big[\nabla_2 h_j(x,u,t)^T + \eta^j(t)^T Q_j(t)\big]\Bigg\}$$

$$- \big[\Pi_i(x,u,\alpha^i,\beta^i) + \Delta(x,u,v,w,\zeta,\eta)\big]\big[\nabla_2 g_i(x,u,t)^T - \delta^i(t)^T N_i(t)\big]\Big\}\Big\}(\bar{u}-u) \geqq 0, \quad \text{for all } t \in [a,b] \text{ and all } \bar{u} \in \mathrm{PWS}^m[a,b]$$

such that $(x,\bar{u}) \in \mathcal{F}$ for some $x \in C^n[a,b]$.

Following the pattern of Theorems 4.4–4.6, one can easily state and prove similar theorems for (P)–(EII).

6. Problems containing square roots of positive semidefinite quadratic forms

In this section we shall briefly discuss a special case of (P) obtained when all the norms are chosen to be the ℓ_2-norm $\|\cdot\|_2$.

Let $k(i) = \ell(i) = m(i) = n(i) = p(j) = q(j) = 2$, $i \in \underline{r}$, $j \in \underline{s}$. In this case, if we let $E_i(t) = K_i(t)^T K_i(t)$, $F_i(t) = L_i(t)^T L_i(t)$, $G_i(t) = M_i(t)^T M_i(t)$, $H_i(t) = N_i(t)^T N_i(t)$, $i \in \underline{r}$, $R_j(t) = P_j(t)^T P_j(t)$ and $S_j(t) = Q_j(t)^T Q_j(t)$, $j \in \underline{s}$, then it is easily seen that for all $i \in \underline{r}$ and $j \in \underline{s}$, $E_i(t)$, $G_i(t)$, and $R_j(t)$ are $n \times n$ symmetric positive semidefinite matrices, $F_i(t)$, $H_i(t)$, and $S_j(t)$ are $m \times m$ symmetric positive semidefinite matrices, and, therefore, the functions $x(t) \to [x(t)^T E_i(t) x(t)]^{1/2}$, $x(t) \to [x(t)^T G_i(t) x(t)]^{1/2}$, and $x(t) \to [x(t)^T R_j(t) x(t)]^{1/2}$ are convex on $\Re^n$, and the functions $u(t) \to [u(t)^T F_i(t) u(t)]^{1/2}$, $u(t) \to [u(t)^T H_i(t) u(t)]^{1/2}$, and $u(t) \to [u(t)^T S_j(t) u(t)]^{1/2}$ are convex on $\Re^m$. With these choices of the norms and matrices, (P), (PI), (P2), and (P3) become

$$(\mathrm{P}^*) \quad \text{Minimize} \left(\frac{\int_a^b \{f_1(x,u,t) + [x^T E_1(t)x]^{1/2} + [u^T F_1(t)u]^{1/2}\}\,dt}{\int_a^b \{g_1(x,u,t) - [x^T G_1(t)x]^{1/2} - [u^T H_1(t)u]^{1/2}\}\,dt}, \ldots, \frac{\int_a^b \{f_r(x,u,t) + [x^T E_r(t)x]^{1/2} + [u^T F_r(t)u]^{1/2}\}\,dt}{\int_a^b \{g_r(x,u,t) - [x^T G_r(t)x]^{1/2} - [u^T H_r(t)u]^{1/2}\}\,dt} \right)^T$$

subject to

$$x(a) = 0, \qquad x(b) = 0, \tag{6.1}$$

$$Dx = A(t)x + B(t)u, \quad t \in [a,b], \tag{6.2}$$

$$h_j(x,u,t) + \big[x^T R_j(t)x\big]^{1/2} + \big[u^T S_j(t)u\big]^{1/2} \leqq 0, \quad t \in [a,b],\ j \in \underline{s},$$

$$x \in C^n[a,b],\ u \in \mathrm{PWS}^m[a,b]; \tag{6.3}$$

$$(\mathrm{P}^*1) \quad \underset{(x,u)\in\mathcal{F}^*}{\text{Minimize}} \left(\int_a^b \big\{f_1\big(x,u,t\big) + \big[x^T E_1(t)x\big]^{1/2} + \big[u^T F_1(t)u\big]^{1/2}\big\}\,dt, \ldots, \int_a^b \big\{f_r(x,u,t) + \big[x^T E_r(t)x\big]^{1/2} + \big[u^T F_r(t)u\big]^{1/2}\big\}\,dt \right)^T,$$

$$(\mathrm{P}^*2)\quad \underset{(x,u)\in\mathcal{F}^*}{\text{Minimize}}\ \frac{\int_a^b\{f_1(x,u,t)+[x^T E_1(t)x]^{1/2}+[u^T F_1(t)u]^{1/2}\}\,dt}{\int_a^b\{g_1(x,u,t)-[x^T G_1(t)x]^{1/2}-[u^T H_1(t)u]^{1/2}\}\,dt},$$

$$(\mathrm{P}^*3)\quad \underset{(x,u)\in\mathcal{F}^*}{\text{Minimize}}\int_a^b\left\{f_1(x,u,t)+\left[x^T E_1(t)x\right]^{1/2}+\left[u^T F_1(t)u\right]^{1/2}\right\}dt,$$

where $\mathcal{F}^*$ is the feasible set of (P^*), that is,

$$\mathcal{F}=\left\{(x,u)\in C^n[a,b]\times \mathrm{PWS}^m[a,b]\text{: (6.1)–(6.3) hold}\right\}.$$

Obviously, all the proper efficiency and duality results established for (P) can readily be specialized and restated for (P^*), $(\mathrm{P}^*\mathrm{l})$, (P^*2), and (P^*3).

7. Concluding remarks

In this paper we have established semiparametric necessary and sufficient conditions characterizing properly efficient solutions of a class of constrained multiobjective fractional optimal control problems with linear dynamics, containing arbitrary norms. Furthermore, using these proper efficiency results as a basis, we have constructed eight semiparametric duality models for this class of problems and have proved appropriate duality theorems.

Constrained optimization problems involving norms arise naturally in many areas of the decision sciences, applied mathematics, and engineering. These problems occur most frequently in location theory, approximation theory, and engineering design. Similarly, mathematical programming problems containing square roots of positive semidefinite quadratic forms have been encountered in stochastic programming, multifacility location problems, and portfolio selection problems, among others. Various types of these problems have been investigated mostly in a finite-dimensional setting and a number of optimality and duality results for them have been published in the related literature. Numerous references pertaining to several aspects of both of the above-mentioned classes of problems are cited in [5].

The proper efficiency and duality criteria developed in this paper and in [6] for (P), (PI), (P2), (P3), (P^*), $(\mathrm{P}*\mathrm{l})$, (P^*2), and (P^*3), improve and generalize a number of existing results in the area of optimal control theory, and provide continuous-time analogues of a great variety of cognate problems and results previously investigated in the areas of finite-dimensional nonlinear, fractional, and multiobjective programming.

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